Math 100:V02 – SOLUTIONS TO WORKSHEET 12
DIFFERENTIAL EQUATIONS

1. Differential equations

(1) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

Solution: Plugging in 2 we have $2^2 = 4$ in the first equation but $2^2 \neq 3 \cdot 2$. Plugging in 3 we have $3^2 \neq 4$ but $3^2 = 3 \cdot 3$. The solutions to the first equations are $\{\pm 2\}$, to the second $\{0, 3\}$.

(2) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

Solution: Plugging in $y = x^2$ into the equations we have $2x \neq x^2$ but $(2x)^2 = 2 \cdot x^2$ is true. Plugging in e^x into the equations we see $e^x = e^x$ but $(e^x)^2 = e^{2x} \neq 4e^x$.

(3) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

Solution: $2t + t^2 - 1 \neq t^2$ but $(2t + 2) + t^2 - 1 = t^2 + 2t + 1$ so only B is a solution. If w is another solution them we have

$$\frac{dw}{dt} + t^2 - 1 = w$$
$$\frac{dz}{dt} + t^2 - 1 = z$$

and subtracting the two equations we get $\frac{d(w-z)}{dt} = w - z$ so $w - z = Ce^t$ and $w(t) = Ce^t + t^2 + 2t + 1$ for any constant t.

(4) Which of the following (if any) is a solution of $\frac{dy}{dx} = \frac{x}{y}$

A. $y = -x$; B. $y = x + 5$ C. $y = \sqrt{x^2 + 5}$

Solution: $\frac{d(-x)}{dx} = -1 = \frac{x}{(-x)}$ but $\frac{d(x+5)}{dx} = 1 \neq \frac{x}{x+5}$ and $\frac{d\sqrt{x^2+5}}{dx} = \frac{2x}{2\sqrt{x^2+5}} = \frac{x}{\sqrt{x^2+5}}$ so only A, C are solutions. for any constant t.

(5) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

Solution: The solutions are $Ce^{1.04t}$ for arbitrary C. The particular solution is $100e^{1.04t}$ dollars.

(6) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

Solution: $w' = \left(\frac{y}{z}\right)' = \frac{y'z - yz'}{z^2} = \frac{ayz - ybz}{z^2} = (a-b)\frac{y}{z} = (a-b)w$ so the equation is $\frac{dw}{dx} = (a-b)w$.

2. Solutions by massaging and ansatze

(7) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

Solution: $(\sin(\omega t))' = \omega \cos \omega t$ so $(\sin(\omega t))'' = -\omega^2 \sin(\omega t)$ so

$$(\sin(\omega t))'' = -4 (\sin(\omega t))$$

iff $\omega^2 = 4$, that is iff $\omega = \pm 2$.

Date: 29/2/2024, Worksheet by Lior Silberman.

(8) (The quantum harmonic oscillator) For which value of the constants A, B (with $B > 0$) does the function $f(x) = Axe^{-Bx^2}$ satisfy $-f'' + x^2 f = 3f$? What if we also insist that $f(1) = 1$?

Solution: $f' = Ae^{-Bx^2} - 2ABx^2e^{-Bx^2}$ so $f'' = -6ABxe^{-Bx^2} + 4AB^2x^3e^{-Bx^2}$ and

$$\begin{aligned} -f'' + x^2 f &= 6ABxe^{-Bx^2} + \left(Ax^3e^{-Bx^2} - 4AB^2x^3e^{-Bx^2}\right) \\ &= 6ABxe^{-Bx^2} + A\left(1 - 4B^2\right)x^3e^{-Bx^2} \end{aligned}$$

so

$$-f'' + x^2 f = \left(6B + (1 - 4B^2)x^2\right) Axe^{-Bx^2}$$

and we get a solution to our equation only if $1 - 4B^2 = 0$ that is if $B = \frac{1}{2}$ (and then $6B = 3$ as desired). Finally the solution has $f'(1) = 1$ if $Ae^{-1/2} = 1$ so $A = e^{1/2}$ and $f(x) = xe^{-\frac{1}{2}(x^2-1)}$.

(9) Consider the equation $\frac{dy}{dt} = a(y - b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by u?
Solution: $u' = y' = a(y - b)' = au$.

(b) What is the general solution for $u(t)$?
Solution: $u(t) = Ce^{at}$ where $C = u(0)$.

(c) What is the general solution for $y(t)$?
Solution: $y(t) = u(t) + b = Ce^{at} + b$.

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?
Solution: $y(t) \xrightarrow[x\to\infty]{} b$ and the convergence is exponential: $y(t) - b$ decays exponentially.

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?
Solution: We have $Ce^{a\cdot 0} + b = y(0)$ so $C = y(0) - b$ and $y(t) = (y(0) - b)\,e^{at} + b$.

(10) Example: *Newton's law of cooling.* Suppose we place an object of temperature $T(0)$ in an environment of temperature T_{env}. It turns out that a good model for the temperature $T(t)$ of the object at time t is

$$\frac{dT}{dt} = -k\left(T - T_{\text{env}}\right)$$

where $k > 0$ is a positive constant.

(a) Suppose $T(t) > T_{\text{env}}$. Is $T'(t)$ positive or negative? What if $T(t) < T_{\text{env}}$? Explain this in words.
Solution: If $T(t) > T_{\text{env}}$ and $T - T_{\text{env}} > 0$ so $-k\left(T - T_{\text{env}}\right) < 0$. In other words, the temperature will decrease. If $T < T_{\text{env}}$ we find $T' > 0$ and the temperature will increase. Either way the temperature tends towards T_{env}.

(b) A body is found at 1:30am and its temperature is measured to be 32.5°C. At 2:30am its temperature is found to be 30.3°C. The temperature of the room in which the body was found is measured to be 20°C and we have no reason to believe the ambient temperature has changed. What was the time of death?
Solution: As we have seen above let $u(t) = T(t) - T_{\text{env}}$ and then the equation says the *temperature difference* decays exponentially: $u'(t) = -ku(t)$ and hence $u(t) = u(0)e^{-kt}$. Measuring time in hours and letting $t = 0$ at 1:30am we have $u(0) = 32.5 - 20 = 12.5$ and $u(1) = 30.3 - 20 = 10.3$. We thus have

$$e^{-k} = \frac{u(1)}{u(0)} = \frac{10.3}{12.5}$$

and hence

$$k = \log\frac{12.5}{10.3}\,.$$

The question asks when $T(t) = 37°\text{C}$, that is when $u(t) = 17$. This reads

$$u(0)e^{-kt} = 17$$

so

$$
\begin{aligned}
t &= \frac{1}{k}\log\frac{u(0)}{17} \approx -1.6\text{h} \\
&= \frac{\log(12.5/17)}{\log(12.5/10.3)} \\
&= -\frac{\log(17/12.5)}{\log(12.5/10.3)} \\
&\approx -1.6\text{h} \approx 95\text{min}
\end{aligned}
$$

(11) A body falling through the air is at height $y(t)$ at time t where $y(t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2} = -g + \kappa\left(\frac{dy}{dt}\right)^2 .$$

Here g is the acceleration due to gravity and κ is the *drag coefficient*.

(a) Write the differential equation satisfied by the velocity $v = \frac{dy}{dt}$.
Solution: We have

$$\frac{dv}{dt} = -g + \kappa v^2 .$$

(b) This differential equation has a *fixed point* (also known as a *steady state*): find the value u (called the "terminal velocity") such that the constant function $v(t) \equiv u$ is a solution.
Solution: If v is constant, $\frac{dv}{dt} = 0$ so we need to solve $\kappa u^2 - g = 0$ that is

$$u = \sqrt{\frac{g}{\kappa}} .$$

(c) Define the *hyperbolic trigonometric* functions $\cosh x = \frac{e^x+e^{-x}}{2}$, $\sinh x = \frac{e^x-e^{-x}}{2}$, and $\tanh x = \frac{\sinh x}{\cosh x}$. Check that $(\cosh x)' = \sinh x$, $(\sinh x)' = \cosh x$ and that $(\tanh x)' = 1 - \tanh^2 x$.
Solution: We have

$$
\begin{aligned}
\frac{d}{dx}\cosh x &= \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh x \\
\frac{d}{dx}\sinh x &= \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh x \\
\frac{d}{dx}\tanh x &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\sinh x)'}{\cosh x} - \frac{\sinh x \cdot (\cosh x)'}{(\cosh x)^2} \\
&= \frac{\cosh x}{\cosh x} - \frac{\sinh x \cdot \sinh x}{(\cosh x)^2} = 1 - \tanh^2 x .
\end{aligned}
$$

(d) Find the values of A, α for which

$$v = -A\tanh\left(\alpha(t-t_0)\right)$$

solves the differential equation.
Solution: We have

$$
\begin{aligned}
\frac{dv}{dt} &= -\alpha A\left(1 - \tanh^2\left(\alpha(t-t_0)\right)\right) \\
&= -\alpha A + \frac{\alpha}{A}v^2
\end{aligned}
$$

so we need $\alpha A = g$ and $\frac{\alpha}{A} = \kappa$. Multiplying the two we have $\alpha^2 = g\kappa$ and dividing the two we get $A^2 = \frac{g}{\kappa}$. We therefore have $A = u$ and that the solution is

$$v(t) = -u\tanh\left(\sqrt{\kappa g}(t-t_0)\right)$$

(e) Show that $\lim_{x\to\infty}\tanh x = 1$ and conclude that $v(t)$ indeed converges to the terminal velocity as $t \to \infty$.