Math 100:V02 – SOLUTIONS TO WORKSHEET 11
INVERSE TRIG; LOGARITHMIC DIFFERENTIATION

1. Logarithmic differentiation

(1) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2+3t\right) =$

Solution: By the chain rule, the derivatives are: $\frac{1}{ax}\cdot a = \frac{1}{x}$ and $\frac{1}{t^2+3t}\cdot(2t+3) = \frac{2t+t}{t^2+3t}$. We can also use the logarithm laws first: $\log(ax) = \log a + \log x$ so $\frac{\mathrm{d}}{\mathrm{d}x}(\log ax) = \frac{\mathrm{d}}{\mathrm{d}x}(\log a) + \frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}$ since $\log a$ is constant if a is. Similarly, $\log(t^2+3t) = \log t + \log(t+3)$ so its derivative is $\frac{1}{t} + \frac{1}{t+3}$.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^2\log(1+x^2) =$ $\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$

Solution: Applying the product rule and then the chain rule we get: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\log(1+x^2)\right) = 2x\log(1+x^2) + x^2\frac{1}{1+x^2}\cdot 2x = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$. Using the quotient rule and the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} = -\frac{1}{\log^2(2+\sin r)}\cdot\frac{1}{2+\sin r}\cdot\cos r = -\frac{\cos r}{(2+\sin r)\log^2(2+\sin r)}\,.$$

(2) (Logarithmic differentiation) differentiate $y = \left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}$.

Solution: We have

$$\begin{aligned}\log y &= \log\left(x^2+1\right) + \log(\sin x) + \log\left(\frac{1}{\sqrt{x^3+3}}\right) + \log\left(e^{\cos x}\right)\\ &= \log\left(x^2+1\right) + \log(\sin x) - \frac{1}{2}\log\left(x^3+3\right) + \cos x\,.\end{aligned}$$

Differentiating with respect to x gives:

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{3x^2}{x^3+3} - \sin x$$

and solving for y' finally gives

$$y' = \left(\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x}{2(x^3+3)} - \sin x\right)\cdot\left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}\,.$$

(3) Differentiate using $\boxed{f' = f\times(\log f)'}$

(a) x^n

Solution: If $y = x^n$ then $\log y = n\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \frac{n}{x}$ so $y' = y\frac{n}{x} = nx^{n-1}$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = x^n\frac{\mathrm{d}}{\mathrm{d}x}(\log(x^n)) = x^n\left(\frac{n}{x}\right) = nx^{n-1}$.

(b) x^x

Solution: If $y = x^x$ then $\log y = x\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \log x + x\cdot\frac{1}{x} = \log x + 1$ so $y' = y(\log x+1) = x^x(\log x+1)$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}(x^x) = x^x\frac{\mathrm{d}}{\mathrm{d}x}(\log(x^x)) = x^x(\log x+1)$.

Solution: We have $x^x = \left(e^{\log x}\right)^x = e^{x\log x}$. Applying the chain rule we now get $(x^x)' = e^{x\log x}(\log x+1) = x^x(\log x+1)$.

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(c) $(\log x)^{\cos x}$

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\log x)^{\cos x} &= (\log x)^{\cos x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\cos x\log(\log x))\\ &= -\sin x\log\log x\,(\log x)^{\cos x} + (\log x)^{\cos x}\cos x\frac{1}{\log x}\frac{1}{x}\\ &= -\sin x\log\log x\,(\log x)^{\cos x} + \cos x\,(\log x)^{\cos x-1}\frac{1}{x}\,.\end{aligned}$$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= y\frac{\mathrm{d}\log y}{\mathrm{d}x} = x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}(\log x\cdot\log x)\\ &= x^{\log x}\left(2\log x\cdot\frac{1}{x}\right) = 2\log x\cdot x^{\log x-1}\,.\end{aligned}$$

(4) Let $f(x) = g(x)^{h(x)}$. Find a formula for f' in terms of g' and h'.

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}f' &= f\cdot(h\log g)'\\ &= f\left(h'\log g + \frac{h}{g}g'\right)\\ &= h\cdot g^{h-1}\cdot g' + g^h\log g\cdot h'\,.\end{aligned}$$

Observe that this is the sum of what we'd get by applying the power law rule and the exponential rule.

2. Inverse trig

(5) (evaluation)

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.

Solution: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. Also $\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9\pi}{11}\right) = \sin\left(\pi - \frac{9\pi}{11}\right) = \sin\left(\frac{2\pi}{11}\right)$ and $\frac{2\pi}{11} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right) = \frac{2\pi}{11}$.

(b) (Final 2015) Simplify $\sin(\arctan 4)$

Solution: Consider the right-angled triangle with sides 4, 1 and hypotenuse $\sqrt{1+4^2} = \sqrt{17}$. Let θ be the angle opposite the side of length 4. Then $\tan\theta = 4$ and $\sin\theta = \frac{4}{\sqrt{17}}$ so $\sin(\arctan 4) = \sin\theta = \frac{4}{\sqrt{17}}$.

(c) Find $\tan(\arccos(0.4))$

Solution: Consider the right-angled triangle with sides 0.4, $\sqrt{1-0.4^2}$ and hypotenuse 1. Let θ be the angle between the side of length 0.4 and the hypotenuse. Then $\cos\theta = \frac{0.4}{1} = 0.4$ and $\tan\theta = \frac{\sqrt{1-0.4^2}}{0.4} = \frac{\sqrt{0.84}}{0.4} = \sqrt{\frac{0.84}{0.16}} = \sqrt{5.25}$.

(6) Let $f(\theta) = \sin^2\theta + \cos^2\theta$. Find $\frac{df}{d\theta}$ without using trigonometric identities. Evaluate $f(0)$ and conclude that $\sin^2\theta + \cos^2\theta = 1$ for all θ.

Solution: By the chain rule $\frac{d}{d\theta}(\sin\theta)^2 = 2\sin\theta\cos\theta$ and $\frac{d}{d\theta}(\cos\theta)^2 = 2\cos\theta(-\sin\theta)$ so

$$\frac{df}{d\theta} = 2\sin\theta\cos\theta - 2\sin\theta\cos\theta = 0\,,$$

It follows that f is constant; since $f(0) = (\sin 0)^2 + (\cos 0)^2 = 1$ we have $f(\theta) = 1$ for all θ, which is the claim.

(7) (Inverse functions)

(a) Suppose $g(x) = e^x$, $f(y) = \log y$. Show that $f(g(x)) = x$ and conclude that $(\log y)' = \frac{1}{y}$.

Solution: $f(g(x)) = \log(e^x) = x$. We then have $f'(e^x) = \frac{1}{g'(x)} = \frac{1}{e^x}$ so $f'(y) = \frac{1}{y}$ for all $y > 0$.

(b) Let $\theta = \arcsin x$. Find $\frac{d\theta}{dx}$. *Hint*: solve for x first.
Solution: We have $x = \sin\theta$ so $1 = \cos\theta \frac{d\theta}{dx}$ so

$$\frac{dx}{d\theta} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}\,.$$

(8) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(2x\right)\right)$
Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arcsin\left(2x\right)\right) = \frac{2}{\sqrt{1-4x^2}}\,.$$

Alternatively, let $\theta = \arcsin 2x$, so that $\sin\theta = 2x$. Differentiating both sides we get

$$\cos\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} = 2$$

so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{2}{\cos\theta} = \frac{2}{\sqrt{1-\sin^2\theta}} = \frac{2}{\sqrt{1-4x^2}}\,.$$

(b) Find the line tangent to $y = \sqrt{1+\left(\arctan(x)\right)^2}$ at the point where $x = 1$.
Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}$, the chain rule gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\left(\arctan(x)\right)^2} &= \frac{1}{2\sqrt{1+\left(\arctan(x)\right)^2}} \cdot 2\arctan(x) \cdot \frac{1}{1+x^2} \\ &= \frac{\arctan x}{(1+x^2)\sqrt{1+\left(\arctan(x)\right)^2}}\,.\end{aligned}$$

Now $\arctan 1 = \frac{\pi}{4}$ so the line is

$$y = \frac{\pi}{8\sqrt{1+\frac{\pi^2}{16}}}\left(x-1\right) + \sqrt{1+\frac{\pi^2}{16}}\,.$$

(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?
Solution: From the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(e^{5x}\right) = \frac{1}{\sqrt{1-e^{10x}}}5e^{5x} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}\,.$$

The function y itself is defined when $-1 \le e^{5x} \le 1$, that is when $5x \le 0$, that is when $x \le 0$. The derivative is defined when $-1 < e^{10x} < 1$, that is when $x < 0$. The point is that since $\sin\theta$ has horizontal tangents at $\pm\frac{\pi}{2}$, $\arcsin x$ has vertical tangents at ± 1.
Solution: We can write the identity as $\sin y = e^{5x}$ and differentiate both sides to get $y'\cos y = 5e^{5x}$ so that

$$y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1-\sin^2 y}} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}\,.$$