Math 100:V02 – SOLUTIONS TO WORKSHEET 8
APPLICATIONS OF THE CHAIN RULE

1. Review

(1) Differentiate
  (a) $e^{\sqrt{\cos x}}$
  Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x}\\ &= e^{\sqrt{\cos x}}\frac{1}{2\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\cos x\\ &= -e^{\sqrt{\cos x}}\frac{\sin x}{2\sqrt{\cos x}}\,.\end{aligned}$$

(2) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.
Solution: Take the logarithm of both sides to get

$$\begin{aligned}\log y &= \log\left(x^{\log x}\right)\\ &= \log x \cdot \log x = (\log x)^2\,.\end{aligned}$$

Differentiating both sides we find

$$\frac{1}{y}\frac{dy}{dx} = 2\left(\log x\right)\frac{1}{x}$$

so solving for the derivatives we find

$$\frac{dy}{dx} = 2y\frac{\log x}{x} = 2x^{\log x - 1}\log x\,.$$

2. Implicit Differentiation

(3) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.
Solution: Differentiating with respect to x we find $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 2$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2+1}{y}$. In particular at the point $(2, 6)$ the slope is $\frac{25}{6}$ and the line is

$$y = \frac{25}{6}(x - 2) + 6\,.$$

(4) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 1)$.
Solution: Differentiating with respect to x we find $y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ along the curve. Setting $x = y = 1$ we find that, at the indicated point,

$$3 + 3\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = 0$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = -1\,.$$

(5) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.
Solution: Differentiating with respect to x we find $y' + \cos y - x\sin y \cdot y' = -\sin x$, so that $y' = -\frac{\sin x + \cos y}{1 - x\sin y} = \frac{\sin x + \cos y}{x\sin y - 1}$. Setting $x = 0$, $y = 1$ we get that at that point $y' = \frac{\cos 1}{-1} = -\cos 1$.

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(6) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

Solution: Differentiating with respect to x we find $5x^4 + 5y^4 y' = 0$, so that $y' = -\frac{x^4}{y^4}$. Differentiating again we find

$$y'' = -\frac{4x^3}{y^4} + \frac{4x^4 y'}{y^5} = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9}\,.$$

3. Related Rates

(5) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $\left(1, \sqrt{3}\right)$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.

Solution: We differentiate along the curve with respect to time, finding

$$2y\frac{\mathrm{d}y}{\mathrm{d}t} = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

Plugging in $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$, $x = 1$, $y = \sqrt{3}$ we find: $2\sqrt{3} = 5\frac{\mathrm{d}x}{\mathrm{d}t}$ so at that time we have

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2\sqrt{3}}{5}\,.}$$

(6) The state of a quantity of gas in a piston must satisfy the *ideal gas law*

$$PV = nRT\,,$$

where P is the pressure, V is the volume, n is the number of moles of gas, T is the (absolute) temperature and R is the ideal gas constant. Suppose $P = 1$atm and $V = 22.4$L. How fast is the pressure of the gas changing when $\frac{dV}{dt} = 2.5\frac{\mathrm{L}}{\mathrm{min}}$, if the expansion is *isothermal*, that is with T held constant?

Solution: We differentiate along the curve with respect to time, finding

$$\frac{dP}{dt}V + P\frac{dV}{dt} = nR\frac{dT}{dt} = 0\,.$$

This gives

$$\begin{aligned}\frac{dP}{dt} &= -\frac{P}{V}\frac{dV}{dt}\\ &= \boxed{-\frac{2.5}{22.4}\frac{\mathrm{atm}}{\mathrm{min}}\,.}\end{aligned}$$

4. Partial derivatives

(7) Returning to the equation $PV = nRT$ now treat the temperature as a *function* of both pressure and volume.

(a) Suppose the volume is constant. What is the rate of change of temperature with respect to pressure?
Solution: We have $T = \frac{PV}{nR}$ which is linear in P so $\frac{\partial T}{\partial P} = \frac{V}{nR}$

(b) Suppose the pressure is constant. What is the rate of change of temperature with respect to pressure?
Solution: For the same reason $\frac{\partial T}{\partial V} = \frac{P}{nR}$

(c) What is the rate of change of the temperaure with respet to the number of moles of gas, pressure and volume being constant?
Solution: Now $T = \frac{PV}{R} \cdot n^{-1}$ so $\frac{\partial T}{\partial n} = -\frac{PV}{n^2 R}$.