

Math 100:V02 – SOLUTIONS TO WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. THE LADDER OF FUNCTIONS

- (1) Classify the following functions into *power laws / power functions* and *exponentials*: $x^3, \pi x^{102}, e^{2x}, c\sqrt{x}, -\frac{8}{x}, 7^x, 8 \cdot 2^x, -\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x}, \frac{9}{x^{7/2}}, x^e, \pi^x, \frac{A}{x^b}$.

Solution: Power laws: $x^3, \pi x^{102}, c\sqrt{x} = cx^{-1/2}, -\frac{8}{x} = -8x^{-1}, \frac{9}{x^{7/2}} = 9x^{-7/2}, x^e, \frac{A}{x^b} = Ax^{-b}$
Exponentials: $e^{2x} = (e^2)^x, 7^x, 8 \cdot 2^x, -\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x} = -\frac{1}{\sqrt{3}} 2^{-x}, \pi^x$.

- (2) Order the following functions from small to large *asymptotically as $x \rightarrow \infty$* :

- (a) $1, \sqrt{x}, x^{-1/2}, x^{1/3}, e^x, x^{-1/3}, 10^6 x^{2024}, e^{-x}, e^{x^2}, \frac{2024}{x^{100}}, 5^x, x$.

Solution: As $x \rightarrow \infty$ we have

$$e^{-x} \ll \frac{2024}{x^{100}} \ll x^{-1/2} \ll x^{-1/3} \ll 1 \ll x^{1/3} \ll x^{1/2} \ll x \ll 10^6 x^{2024} \ll e^x \ll 5^x \ll e^{x^2}$$

- (b) Extra: add in $\log x, e^{\sqrt{x}}, (\log x)^2, \log \log x, \frac{1}{\log x}$.

Solution: As $x \rightarrow \infty$ we have

$$e^{-x} \ll \frac{2024}{x^{100}} \ll x^{-1/2} \ll x^{-1/3} \ll \frac{1}{\log x} \ll \log \log x \ll \log x \ll (\log x)^2 \ll 1 \ll x^{1/3} \ll x^{1/2} \ll x \ll 10^6 x^{2024} \ll e^{\sqrt{x}} \ll e^x \ll e^{x^2}$$

- (c) Repeat (a), this time as $x \rightarrow 0^+$.

Solution: As $x \rightarrow 0$ we have

$$10^6 x^{2024} \ll x \ll x^{1/2} \ll x^{1/3} \ll 1 \sim e^x \sim e^{-x} \sim e^{x^2} \sim 5^x \ll x^{-1/3} \ll x^{-1/2} \ll \frac{2024}{x^{100}}$$

2. ASYMPTOTICS: SIMPLE EXPRESSIONS

- (3) How does the each expression behave when x is large? small? what is x is large but negative? Sketch a plot

- (a) $1 - x^2 + x^4$ (“Mexican hat potential”)

Solution: When x is large (positive or negative), $x^4 \gg x^2 \gg 7$ so $7 + x^2 + x^4 \sim x^4$ while when x is small, $7 \gg x^2 \gg x^4$ so $7 + x^2 + x^4 \sim 7$.

- (b) $ax^3 - bx^5$ ($a, b > 0$)

Solution: When x is very large, x^5 dominates x^3 so $ax^3 - bx^5 \sim -ax^5$ (which is negative for x positive, positive for x negative!). When x is very small (close to zero), x^3 dominates (is bigger than x^5 though both are very small) and $ax^3 - bx^5 \sim ax^3$.

- (c) $e^x - x^4$

Solution: When $x \rightarrow \infty$ is very large, $e^x \gg x^4$ so $e^x - x^4 \sim e^x$. Near we have $e^x \sim 1 \gg x^4$, so $e^x - x^4 \sim 1$. Finally when x is large but negative ($x \rightarrow -\infty$) we have that e^x decays while x^4 grows, so $e^x \ll x^4$ and $e^x - x^4 \sim -x^4$.

- (d) Wages in some country grow at 2% a year (so the wage of a typical worker has the form $A \cdot (1.02)^t$ where t is measured in years and A is the wage today). The cost of healthcare grows at 4% a year (so the healthcare costs of a typical worker have the form $B \cdot (1.04)^t$ where B is the cost today). Suppose that today’s workers can afford their healthcare (A is much bigger than B). Will that be always true? Why or why not?

Solution: Asymptotically $(1.04)^t$ will dominate 1.02^t for large t , so eventually our assumptions must break down.

- (e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t.$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?

Solution: When t is large, $(0.98)^t$ is actually decaying so this strain will disappear. On the other hand since $1.1 > 1.05$ over time 1.1^t will be much bigger than

- (4) The (attractive) interaction between two hadrons (say protons) due to the strong nuclear force can be modeled by the *Yukawa potential* $V_Y(r) = -g^2 \frac{e^{-\alpha m r}}{r}$ where r is the separation between the particles, and g, α, m are positive constants. The electrical repulsion between two protons is described by the Coulomb potential $V_C(r) = kq^2 \frac{1}{r}$ where k, q are also positive constants. Which interaction will dominate for large distances? Will the net interaction be attractive or repulsive? Note that g^2 is much larger than kq^2 .

Solution: At large distances the exponentially decaying factor will suppress the strong interaction, making the electrical interaction dominate. This is why nuclear fusion requires such high temperatures: we need to get the protons **really close** to each other for the strong force to take over, and this requires them moving very fast or the electrical repulsion will keep them apart.

3. ASYMPTOTICS OF COMPLICATED EXPRESSIONS

- (5) Describe the following expressions in words

(a) $e^{|x-5|^3}$

Solution: This is the exponential, of the cube, of the absolute value, of $x - 5$.

(b) $\frac{1+x}{1+2x-x^2}$

Solution: This is the ratio of (the sum of 1 and x) and (the sum of 1, $2x$, and $-x^2$).

(c) $\frac{e^x + A \sin x}{e^x - x^2}$

Solution: This is the ratio of (the sum of e^x and the product of A and $\sin x$) and (the difference of e^x and x^2).

(d) $\frac{Ae^{rt} + Be^{-st}}{t + t^2}$ where $r, s > 0$ and $A, B \neq 0$.

Solution: This is the sum of A times the exponential of r times t and B times the exponential of $-s$ times t , all divided by the sum of t and t^2 .

- (6) For each of the functions in (a),(b),(c),(d) determine its asymptotics near 0 and near $+\infty$.

(a)

Solution: (a) For x close to 0, $x - 5 \sim -5$ so $|x - 5| \sim 5$ so $|x - 5|^3 \sim 125$ so $e^{|x-5|^3} \sim e^{125}$. For x very large $x - 5 \sim x$ and since x is positive $|x - 5| \sim |x| = x$ so $|x - 5|^3 \sim x^3$. $e^{|x-5|^3}$ therefore grows roughly like e^{x^3} (in truth e^{x^3} is actually much bigger than $e^{(x-5)^3}$ - the ratio is on the scale of e^{15x^2} - but our expression captures the gist of the growth pattern).

(b)

Solution: (d) As $x \rightarrow 0$ x, x^2 are negligible next to the 1 so $\frac{1+x}{1+2x-x^2} \sim \frac{1}{1} = 1$. As $x \rightarrow \infty$ x dominates 1 so $x+1 \sim x$ and x^2 dominates x , 1 so $1+2x-x^2 \sim -x^2$. Thus $\frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} = -\frac{1}{x}$ - in other words the whole expression decays roughly like $\frac{1}{x}$.

(c)

Solution: (c) For x near 0 we have $e^x \sim e^0 = 1$ and $\sin x \rightarrow 0$ (we'll later learn that $\sin x \sim x$ near 0) so $e^x + A \sin x \sim 1$ near 0. Similarly $x^2 \sim 0$ so $e^x - x^2 \sim 1$ and we have $\frac{e^x + A \sin x}{e^x - x^2} \sim \frac{1}{1} = 1$. For large x we have $|\sin x| \leq 1$ so $A \sin x$ is much smaller than e^x and $e^x + A \sin x \sim e^x$. Similarly e^x dominates any polynomial including x^2 and we have $e^x - x^2 \sim e^x$. Thus at infinity $\frac{e^x + A \sin x}{e^x - x^2} \sim \frac{e^x}{e^x} = 1$.

(d)

Solution: (d) As $t \rightarrow 0$ we have $t^2 \ll t$ so $t + t^2 \sim t$. $e^{rt} \sim e^0 \sim e^{-st}$ so

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{A + B}{t}.$$

As $t \rightarrow \infty$, $t^2 \gg t$ while $e^{rt} \gg e^{-st}$ (growing exponential dominates the decaying one!). Thus

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{Ae^{rt}}{t^2}.$$

Conversely as $t \rightarrow -\infty$ we have $e^{-st} \gg e^{rt}$ so

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{Be^{-st}}{t^2}.$$