Math 100, lecture 23, 4/4/2024

Today: Graphs of multivariable functions

Next time: Optimization

Have f function of (x,y). Graph of f is the surface

$$z = f(x,y)$$

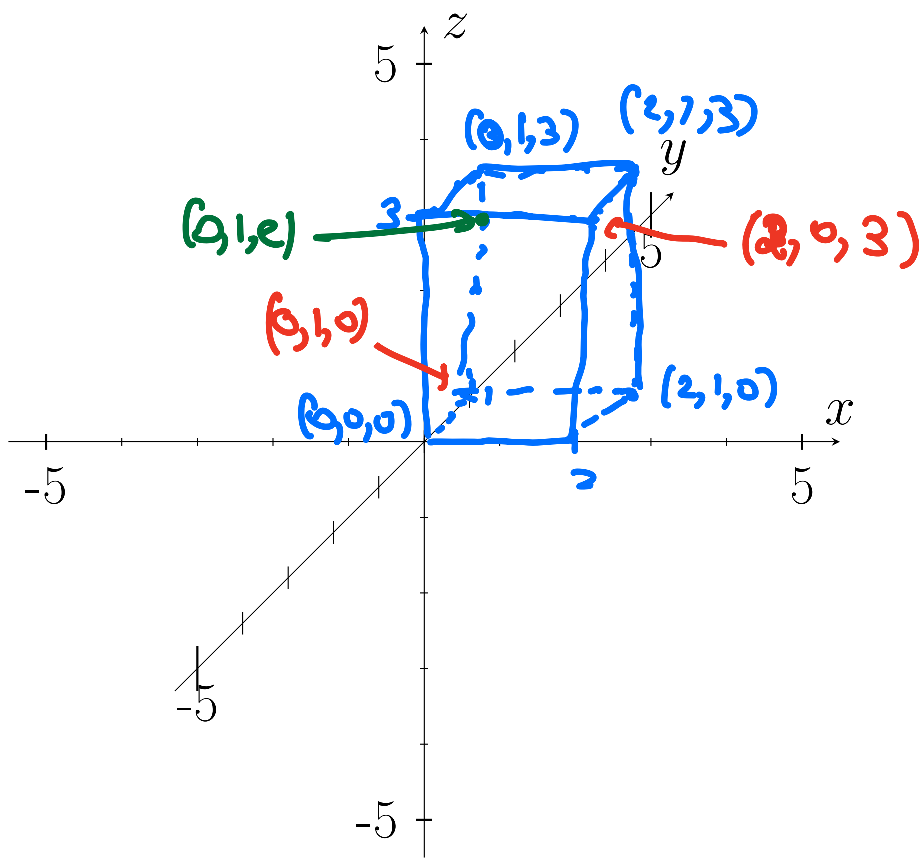
$\Rightarrow$ (1) need to understand 3d space: projection

(2) topographical intuition/terminology

(3) combine with calculus

Math 100:V02 – WORKSHEET 17
MULTIVARIABLE DIFFERENTIATION

1. Plotting in three dimensions



(1) ⋆ Plot the points $(2, 1, 3)$, $(-2, 2, 2)$ on the axes provided.

$f(0,-1) = e$, $f(1,2) = e^5$

(2) Let $f(x, y) = e^{x^2+y^2}$.

(a) ⋆ What are $f(0, -1)$? $f(1, 2)$? Plot the point $(0,1,e) = (0, 1, f(0, 1))$ on the axes provided.

Date: 4/4/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) ⋆ What is the *domain* of f (that is: for what (x, y) values does f make sense?

whole plane

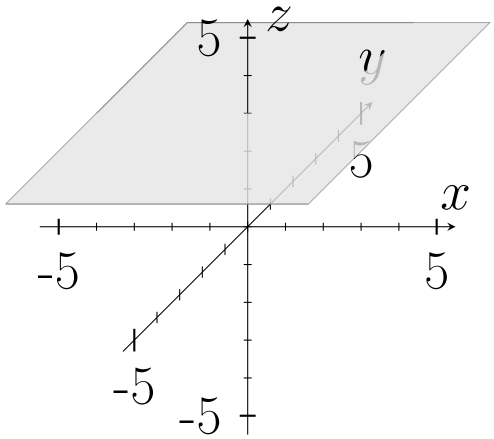
(c) ⋆ What is the *range* of f (that is: what values does it take)?

x^2+y^2 takes all values in $[0,\infty)$ so $e^{x^2+y^2}$ takes all values in $[1,\infty)$ $(e^0=1)$

(3) ⋆⋆ What would the graph of $z = \sqrt{1 - x^2 - y^2}$ look like?

points on graph have $z^2 = 1 - x^2 - y^2$ so $x^2+y^2+z^2 = 1$ so graph is on sphere, also $z \geq 0$ so set hemisphere.

(4) ⋆ Which plane is this?



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

2. Partial derivatives

(5)(a) $\star$ Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

(b) $\star$ Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

(c) $\star$ Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

$\frac{\partial f}{\partial x} = 4x$

(d) $\star$ What is $\frac{\partial f}{\partial y}$? $\frac{\partial f}{\partial y} = -2y$.

(6) Find the partial derivatives with respect to both x, y of

(a) $\star\ g(x,y) = 3y^2 \sin(x+3)$

$$\frac{\partial g}{\partial x} = 3y^2 \cos(x+3)$$

$$\frac{\partial g}{\partial y} = 6y \sin(x+3)$$

(b) $\star\ h(x,y) = ye^{Axy} + B$

$$\frac{\partial h}{\partial x} = Ay^2 e^{Axy} \quad ; \quad \frac{\partial h}{\partial y} = e^{Axy} + Axy\, e^{Axy}$$
$$= (1 + Axy) e^{Axy}$$

(7) The the gravitational *potential* due to a point mass M (equivalently the electrical potential due to a point charge M) is given by the formula $U(x,y,z) = -\frac{GM}{r}$ where $r = \sqrt{x^2+y^2+z^2}$. Here G is the universal gravitational constant (equivalently G is the Coulomb constant).

(a) $\star$ The x-component of the field is given by the formula $F_x(x,y,z) = -\frac{\partial U}{\partial x}$. Find F_x

$$-\frac{\partial U}{\partial x} = -\frac{dU}{dr}\cdot\frac{\partial r}{\partial x} = -\frac{GM}{r^2}\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x = -\frac{GM}{r^3}\cdot x$$

chain rule

(b) $\star$ The magnitude of the field is given by $\left|\vec{F}\right| = \sqrt{F_x^2+F_y^2+F_z^2}$. How does it decay as a function of r?

$$\sqrt{\left(\frac{GM}{r^3}x\right)^2+\left(\frac{GM}{r^3}y\right)^2+\left(\frac{GM}{r^3}z\right)^2} = \frac{GM}{r^3}\sqrt{x^2+y^2+z^2} = \frac{GM}{r^2}$$

(8) The *entropy* of an ideal gas of N molecules at temperature T and volume V is

$$S(N,V,T) = Nk\log\left[\frac{VT^{1/(\gamma-1)}}{N\Phi}\right].$$

where k is *Boltzmann's constant* and γ, Φ are constants that depend on the gas.

(a) $\star$ Find the *heat capacity at constant volume* $C_V = T\frac{\partial S}{\partial T}$.

$$S = Nk\log V + \frac{Nk}{\gamma-1}\log T - Nk\log(N\Phi)$$

$$\text{so } \left(\frac{\partial S}{\partial T}\right)_{N,V} = \frac{Nk}{\gamma-1}\frac{1}{T} \quad \text{so } C_V = \frac{Nk}{\gamma-1}.$$

(b) $\star\star\star$ Using the relation ("ideal gas law") $PV = NkT$ write S as a function of N, P, T instead. Differentiating with respect to T **while keeping** P **constant** determine the *heat capacity at constant pressure* $C_P = T\frac{\partial S}{\partial T}$.

$$V = \frac{NkT}{P} \quad \text{so } S = Nk\log\left(\frac{\cancel{N}kT\cdot T^{1/(\gamma-1)}}{P\cancel{N}\Phi}\right)$$

$$= Nk\log\frac{k}{P\Phi} + Nk\left(1+\frac{1}{\gamma-1}\right)\log T$$

$$\text{so } \left(\frac{\partial S}{\partial T}\right)_{P,N} = Nk\cdot\frac{\gamma}{\gamma-1}\cdot\frac{1}{T}, \quad C_P = \frac{Nk\cdot\gamma}{\gamma-1}.$$

(9) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f$. Evaluate:

(a) $\star\ h_{xx} = \frac{\partial^2 h}{\partial x^2} =$

$\frac{\partial h}{\partial x} = Ay^2 e^{Axy}$; $\frac{\partial h}{\partial y} = (1 + Axy) e^{Axy}$

$\Rightarrow \ h_{xx} = A^2 y^3 e^{Axy}$

(b) $\star\ h_{xy} = \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}(h_x) = 2Ay\, e^{Axy} + A^2 x y^2 e^{Axy}$

$= (2Ay + A^2 x y^2) e^{Axy}$

(c) $\star\ h_{yx} = \frac{\partial^2 h}{\partial x \partial y} = Ay\, e^{Axy} + (1 + Axy) Ay \cdot e^{Axy}$

$= (2Ay + A^2 x y^2) e^{Axy}$

(d) $\star\ h_{yy} = \frac{\partial^2 h}{\partial y^2} =$

(11) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street
(say oriented toward the south), and let the y axis run across the street. Let $z = z(x, y)$ denote the height of
the street surface above sea level.

(a) $\star$ What does $\frac{\partial z}{\partial y} = 0$ say about the street?

street is level

(b) $\star$ What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

street has a 15% grade

(c) $\star$ You want to follow the street downhill. Which way should you go?

toward negative x-axis

(d) The intersection of Health Sciences Mall and Agronomy Road is a local maximum.
What does that say about $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ there?

Both have to be zero.