Math 100, lecture 19, 19/3/2024

Last time: Optimization examples.

Today: Numerical methods, Euler scheme.

Problem: Given ODE $y' = f(y)$. Want solution $y(x)$

Earlier: wanted analytic solutions (= formulas)

today: want numerical solution (= compute values)

Idea: Linear approximation

Input: ① ODE $y' = f(y;x)$; ② interval $[a,b]$

③ initial value $y(a) = y_0$

Output: some list of values $x_i \in [a,b]$, y_i so that y_i is "close" to $y(x_i)$.

Question: Make guess for $y(b)$?

Know: $y(a) = y_0$, $y'(a) = f(y(a); a) = f(y_0; a)$

so $$y(b) \approx y(a) + y'(a)\cdot(b-a) \approx y_0 + f(y_0; a)\cdot(b-a)$$

Example: $y' = y^2 + x$, say $y(0)=1$, approx $y(1)$

Then $y'(0) = 1^2 + 0 = 1$ so $y(1) \approx 1 + 1\,(1-0) = 2.$

Basic idea: guess for $y(x_0)$ leads to guess for $y(x_1)$ via linear approx (compute $y'(x_0)$ from DE).

Second idea: intermediate points

Continue with $y' = y^2 + x$, $y(0)=1$ working on $[0,1]$

Approximate $y(\frac{1}{2}) \approx 1 + 1 \cdot \frac{1}{2} = \frac{3}{2}$

Use this to approximate $y(1)$. Have $y'(\frac{1}{2}) \approx (\frac{3}{2})^2 + \frac{1}{2} = \frac{11}{4}$.

so

$$y(1) \approx y(\tfrac{1}{2}) + y'(\tfrac{1}{2}) \cdot (1 - \tfrac{1}{2})$$
$$\approx \frac{3}{2} + \frac{11}{4} \cdot \frac{1}{2} = \frac{23}{8} = 3 - \frac{1}{8}$$

Try partition $[0,1]$ at $0, \frac{1}{3}, \frac{2}{3}, 1$.

$y(0)=1$, $y'(0)=1$, so $y(\frac{1}{3}) \approx 1 + 1 \cdot \frac{1}{3} = 4/3$

$y(\frac{1}{3}) \approx \frac{4}{3}$, $y'(\frac{1}{3}) \approx (\frac{4}{3})^2 + \frac{1}{3} = \frac{19}{9}$, so $y(\frac{2}{3}) \approx \frac{4}{3} + \frac{19}{9} \cdot \frac{1}{3} = \frac{55}{27}$

$y(\frac{2}{3}) \approx \frac{55}{27}$, so $y'(\frac{2}{3}) \approx (\frac{55}{27})^2 + \frac{2}{3} \approx \frac{3511}{729}$, so $y(1) \approx \frac{55}{27} + \frac{3511}{729} \cdot \frac{1}{3} \approx 3.46$

Summary: Euler scheme

for $y' = f(y;x)$, on $[a,b]$ with $y(a) = y_0$

Choose number n of intermediate points

$\Leftrightarrow$ a step size $h = \frac{b-a}{n}$

Get points $x_0 = a$, $x_1 = x_0 + h = a + h$, $x_2 = x_1 + h = x_0 + 2h$, ..

in general $x_i = a + ih$

From y_0 get $y_1 = y_0 + f(y_0; x_0) \cdot h \approx y(x_1)$

then $y_2 = y_1 + f(y_1; x_1) \cdot h \approx y(x_2)$

$y_3 = y_2 + f(y_2; x_2) h$

$\vdots$

$y_{i+1} = y_i + f(y_i; x_i) h$

$\vdots$

Spreadsheet implementation:

	A	B
1	x_0	y_0
2	A1+h	B1 + f(B1; A1)·h
3	A2+h	
i		

copy formulae in A2, B2 down columns