Math 100, Lecture 17, 12/3/2024

Last time: Related Rates

Problem-solving:
(0) read question, diagram (if possible)
(1) name variables
(2) relations between variables
$\Rightarrow$ create objective function (with domain)
(3) calculus
(4) endgame = answer question

Today: Optimization
same idea, different calculus steps
(i) optimization of functions
(ii) optimization problems.

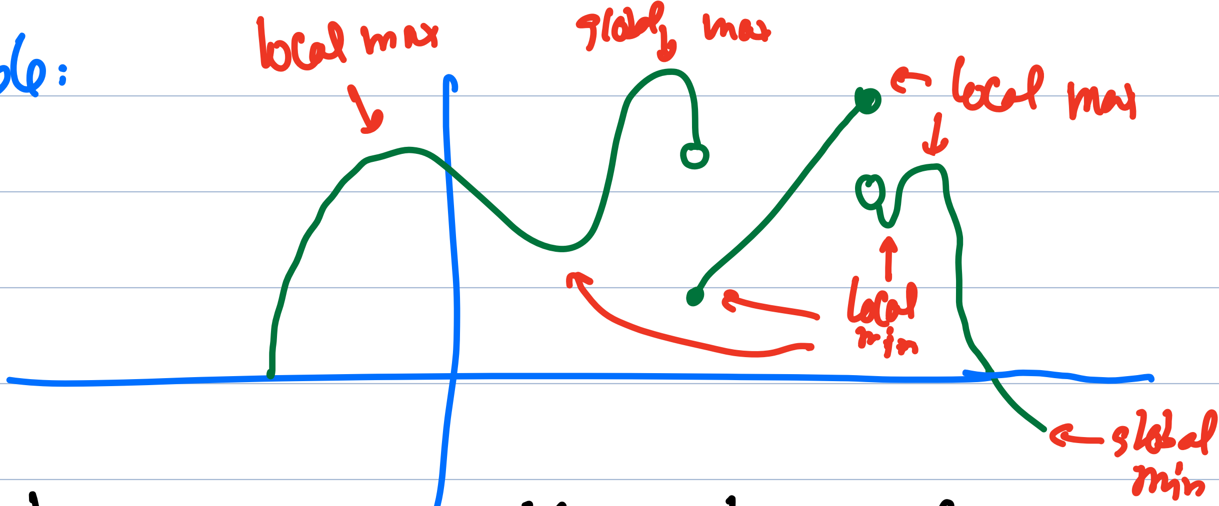
Let f be a function defined on some interval I.

Def: Say f has a local maximum at some interior point $x_0 \in I$ if for all $x \in I$ close enough to x_0, $f(x_0) \geq f(x)$

Say f has a global maximum at $x_0 \in I$, if for all $x \in I$
$f(x_0) \geq f(x)$

(analogous def'n for local/global minimum)

Example:



<u>Optimization</u>: Finding global extrema (values or points)

<u>Calculus</u>: "closed interval method"

<u>Def</u>: A point $(x_0, f(x_0))$ on graph, x_0 in interior of I, is a:
(1) critical point if $f'(x_0) = 0$
(2) singular point if $f'(x_0)$ does not exist

<u>Theorem</u>: Suppose $I = [a, b]$ is closed and bounded. Then the global extrema of f (*) can only occur at:
(1) critical points; (2) singular points; (3) endpoints $(a, f(a)), (b, f(b))$

(*) <u>exist</u> if f is cts and ...

<u>Remark</u>: If f is discontinuous or I not closed or not bounded, need extra arguments about <u>existence</u> of max / min. Often use asymptotics

Math 100:V02 – WORKSHEET 15
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

f is continuous (defined by formula), $[-5,5]$ is closed so closed interval method applies.

$f'(x) = 4x^3 - 8x = 4x(x^2-2)$ so critical points at $(0,4)$, $(\pm\sqrt{2}, 0)$ (or over $0, \sqrt{2}, -\sqrt{2}$)

At ends $f(\pm 5) = 529$.

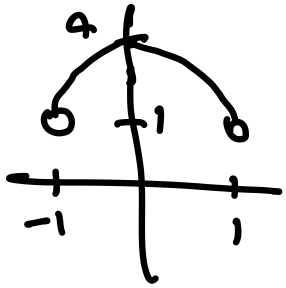
$\Rightarrow$ max = 529, min = 0.

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

$f(\pm 1) = 1$ on $[-1,1]$ only critical pt is $(0,4)$ ($\sqrt{2} > 1$). So max = 4, min = 1.

Date: 12/3/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

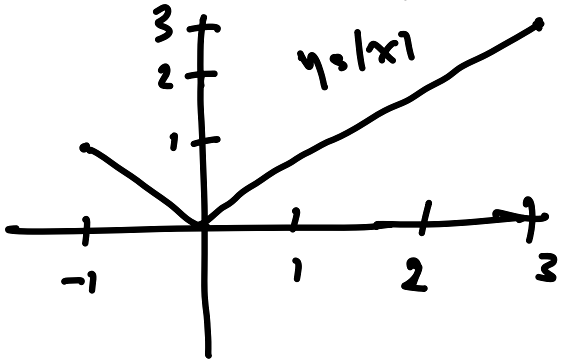
(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.



$f(0) = 4$ global max.

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

(2) Let $f(x) = |x|$. Find the absolute minimum and maximum of f on the interval $[-1, 3]$.



So max = 3 at x=3

min = 0 at x=0

x=0 is a singular point

(3) Find the global extrema (if any) of $f(x) = \frac{1}{x}$ on the intervals $(0, 5)$ and $[1, 4]$.

2. Optimization problems

(4) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

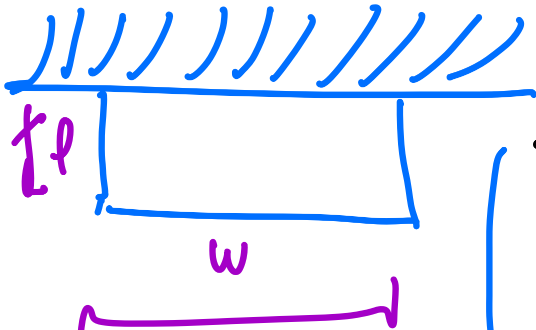
(a) What is the range of r values that makes sense?

$\{r>0\} = (0,\infty)$ since $V(0)$ undefined, distances are non-negative.

(b) Physical systems tend to settle into a state of least energy. Find the minimum of this potential.

(c) Expand the potential to second order about the minimum.

(5) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?



Let l, w be the length & width of the fence in metres

enclosed area = A.

Then: $A = l \cdot w$

$2l + w = 100$ m

so $w = 100 - 2l$

so $A = l(100 - 2l)$

for $0 \leq l \leq 50$ $w \geq 0$

(include degenerate rectangles at $l = 0$, $l = 50$)

$A'(l) = 100 - 2l - 2l$

$= 100 - 4l$

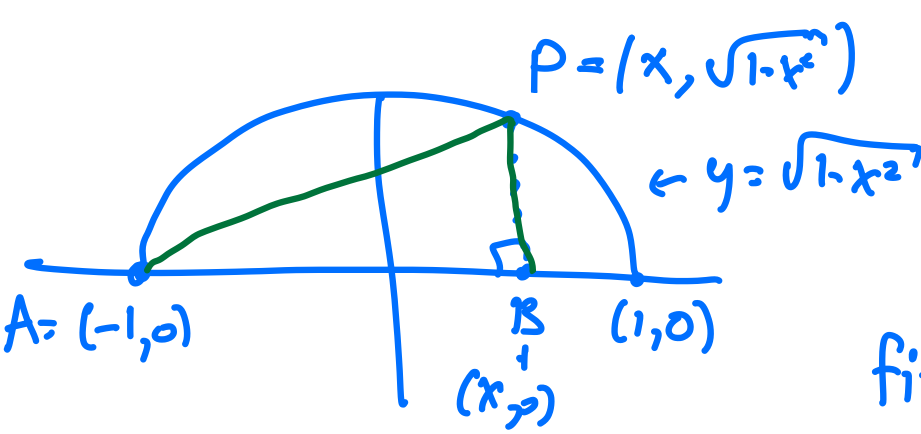
so critical pt over $l_0 = 25$

$A(25) = 25 \cdot 50 = 1250$

$A(0) = A(50) = 0$

The largest area possible is then 1250 m^2.

(6) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?



$\Rightarrow$ triangle has area

$$A = \frac{1}{2}(1+x)\sqrt{1-x^2}$$

find max on $[-1,1]$