Math 100, lecture 11, 13/2/2024

Can see me in ORCH 3009 today 9:30-10:00
11:30-12:00

Ask on $\boxed{\text{Piazza}}$

Last time: Taylor expansion

Understand function f near point a by creating polynomial

$$T_n(x) = C_0 + C_1(x-a) + \dots + C_n(x-a)^n$$

where

$$\boxed{C_k = \frac{f^{(k)}(a)}{k!}} \; ; \; k! = 1\cdot 2\cdot 3 \cdots k$$

$$(0! = 1! = 1)$$

such that

chose c_k s.t. $T_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, 2, \dots, n$

Examples:

$$e^x = 1 + x + x^2/2! + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^k}{k!} + \dots$$

exponential series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

geometric series

(holds if $-1 < x < 1$)

Given f, can find c_k by differentiation; given T_n, can read off c_k

Today: Actually expanding functions

2. New expansions from old

Near $u=0$:	$\frac{1}{1-u} = 1+u+u^2+u^3+u^4\cdots$	$\exp u = 1+\frac{1}{1!}u+\frac{1}{2!}u^2+\frac{1}{3!}u^3+\frac{1}{4!}u^4+\cdots$

(8) ⋆ (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x=0$.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

(either from memory or by diff)

So $\sin\theta \approx \theta - \theta^3/6$ to 3rd order in θ

So $\sin(0.01) \approx 0.01 - (0.01)^3/6$

Also $(x+1)\sin x \approx (1+x)(x - x^3/6) = x + x^2 - x^3/6 - x^4/6 \approx x + x^2 - x^3/6$ correct to 3rd order

(9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x=4$.

let $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$

$f(4) = 2$; $f^{(1)}(4) = \frac{1}{4}$; $f^{(2)}(4) = -\frac{1}{32}$, $f^{(3)}(4) = 3/256$

So $\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$ to 3rd order

$\frac{1}{4}x = \frac{1}{4}(4 + (x-4)) = 1 + \frac{1}{4}(x-4)$ | So $\sqrt{x} - \frac{1}{4}x \approx 1 - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

$e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24}$ to 4th order in u

$e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8$ to 8th order in x

$u = x^2$

$\frac{1}{1-v} \approx 1 + v + v^2$; $v = -x^3$

So $\frac{1}{1+x^3} \approx 1 - x^3 + x^6$ to 8th order in x

So $e^{x^2} - \frac{1}{1+x^3} \approx x^2 + x^3 + \frac{1}{2}x^4 - \frac{5}{6}x^6 + \frac{1}{24}x^8$ to 8th order

$f^{(6)}(0) = 6! \cdot (-5/6) = -600.$

(11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about $x=0$.

to 4th order $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$

So $\cos 3x \approx 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$

So $\frac{1}{\cos 3x} \approx \frac{1}{1 - \frac{9}{2}x^2 + \frac{27}{8}x^4} \approx \frac{1}{1 - \left(\frac{9}{2}x^2 - \frac{27}{8}x^4\right)}$

$\frac{1}{1-u} \approx 1 + u + u^2 + \dots$

$\approx 1 + \left(\frac{9}{2}x^2 - \frac{27}{8}x^4\right) + (\quad)^2$

$\approx 1 + \frac{9}{2}x^2 - \frac{27}{8}x^4 + \frac{81}{4}x^4$

$= 1 + \frac{9}{2}x^2 + \frac{27}{2}x^4.$

other terms have order ≥ 6

(12) (Change of variable/rebasing polynomials)

(a) Find the Taylor expansion of the polynomial $x^3 - x$ about $a=1$ using the identity $x = 1 + (x-1)$.