Math 100:V02
Problem Set 2: Euler's method

Due: Friday April 12th, 2024

Instructions

- Please submit *typeset* solutions through Canvas.
- Solutions must be written in complete English sentences: it's not enough to write a sequence of formulas.
- Do not hesitate to ask for help, whether in-person or on Piazza.

Problem We will examine the following differential equation. In question 1 we will examine the solution qualitatively. In question 2 we will compute them numerically.

$$y' = \left(1 + \frac{1}{2}\sin(2\pi t)\right) y + \sqrt{y} - \frac{1}{3}y^2 \tag{1}$$

1. Let $y(t)$ be a solution to Equation 1.
   (a) What are the asymptotics of $\left(1 + \frac{1}{2}\sin(2\pi t)\right) y + \sqrt{y} - \frac{1}{3}y^2$ as $y \to 0^+$ and $y \to \infty$?
   (b) What is the sign of the right-hand-side of the equation when y is small? When y is large? You may use the fact that $1 - \frac{1}{2} > 0$.
   (c) [hard] explain why, regardless of the initial value $y(0)$ (assumed positive), solutions to the equation will eventually oscillate in some finite band.
2. We will now solve the equation numerically on an interval $[0, b]$ using an Euler scheme with n subintervals, so let $h = b/n$. The ith time will then by $t_i = ih = ib/n$.
   (a) Let $y(t)$ solve 1 and let y_i be the approximation to $y(t_i)$. Write down the Euler scheme approximation to y_{i+1} in terms of y_i and t_i.
   (b) Write a computer program (or spreadsheet program) to approximate the solution with $y(0) = 1$ on the interval $[0, 15]$ using at least 1000 points. Attach a plot of your solution.
   (c) Explain how the plot confirms the prediction from 1(c).

Extra practice (not for submission) Consider instead

$$y' = \left(1 + \frac{1}{2}\sin(2\pi t)\right) y + \sqrt{y} \tag{2}$$

(3) Let $y(t)$ be a solution to Equation 2.
   (a) For which y values does the equation make sense?
   (b) Suppose $y(t) > 0$. What is the sign of $y'(t)$? You may use the fact that $1 - \frac{1}{2} > 0$.
   (c) We now know that $y(t)$ will increase forever, so suppose $y(t)$ is very large. Which of the two terms on the right-hand-side of the equation dominates?
   (d) Suppose we wanted to compare the solution $y(t)$ to a solution of the equation $z' = rz$. Which average growth rate r should we use?

(4) We will now solve the equation numerically as in question (2) above.

(a) Let $y(t)$ solve 2 and let y_i be the approximation to $y(t_i)$. Write down the Euler scheme approximation to y_{i+1} in terms of y_i and t_i.

(b) Write a computer program (or spreadsheet program) to approximate the solution with $y(0) = 1$ on the interval $[0, 15]$. Attach a plot of your solution.

(c) What is your estimate of $y(15)$ to two significant digits? (e.g. write $2,534.7$ as 2.5×10^3)

(d) Plot the ratio $y(t)/e^t$ for your solution (i.e. the points $(t_i, y_i e^{-t_i})$). Explain how this plot relates to our answer from 3(d).