Math 100A, Lecture 14

Last time: **Multivariable Optimization**

① $f(x,y,..)$ on closed, bounded domain has max, min (else need arguments)

② ~~at~~ the interior max/min ~~at~~ critical or singular point. If in

③ If on boundary, on each segment have a subsidiary optimization problem

④ take overall largest/smallest value

Today: Review

① Q: suppose ~~(x,y)~~ is a critical point of f.

How do we tell $(x,y,f(x,y))$ of it's a local max/min/saddle/none of these

A: Sometimes clear from algebra

x^2+y^2 has a min at $(0,0)$

$7x^2-3y^2$ " " saddle point.

sometimes clear because point is global max/min in some domain

(also there exists a 2^{nd} derivative test, but it's not included in MATH 100)

② $f(x,y) = x^2y + y^3 - 12y$ find critical pts

$\frac{\partial f}{\partial x} = 2xy$; $\frac{\partial f}{\partial y} = x^2 + 3y^2 - 12$

so critical pts at $\begin{cases} 2xy = 0 \\ x^2 + 3y^2 = 12 \end{cases}$

1st equation $\Rightarrow$ $x=0$ or $y=0$. If $x=0$ $y = \pm 2$

$y=0$ $x = \pm\sqrt{12}$

so critical pts over $(0,2)$, $(0,-2)$, $(2\sqrt{3}, 0)$, $(-2\sqrt{3}, 0)$

Near pts where $x=0$, $(0,2)$

have

$f(x,y) = \cancel{x^2(y+2)}\ 2x^2 + x^2(y-2) + (2+(y-2))^3 - 12(y-2) - 24 + 8$

$= \underset{\text{0'th order}}{-16} + \underbrace{2x^2 + 6(y-2)^2}_{\text{2nd order}} + \underbrace{x^2(y-2) + (y-2)^3}_{\text{3rd order}}$

local min at $(0,2)$

near $(0,-2)$

$f(x,y) = -2x^2 + (y+2-2)^3 - 12(y+2-2) + (y+2)x^2$

$= -2x^2 + (y+2)^3 - 6(y+2)^2 + 12(y+2) - 8 - 12(y+2) + 24 + (y+2)x^2$

$= 16 - \underbrace{2x^2 - 6(y+2)^2}_{\text{local max at } (0,-2)} + (y+2)^3 + (y+2)x^2$

Near $\pm 2\sqrt{3}$

$$f(x,y) = (x \mp 2\sqrt{3} \pm 2\sqrt{3})^2 y + y^3 - 12y$$

$$= \cancel{12y} \pm 4\sqrt{3}(x \mp 2\sqrt{3})y + (x \mp 2\sqrt{3})^2 y + y^3 - \cancel{12y}$$

$$= \underbrace{4\sqrt{3}(x \mp 2\sqrt{3})y}_{\text{quadratic}} + \underbrace{(x \mp 2\sqrt{3})^2 y + y^3}_{\text{cubic}}$$

saddle point

We found a critical pt over $(0,2)$

Isn't $f(0,2)$ a number?

Yes: $f(0,2) = -16$

But: We wanted f near $(0,2)$, i.e. in terms of $x, y-2$

(Compare: Taylor expansion of e^y about $y=2$ is $e^2\left(1 + (y-2) + \frac{(y-2)^2}{2} + \frac{(y-2)^3}{3!} + \dots\right)$)

Or: say $x = 0 + k$
$y = 2 + h$

Get $f(k, 2+h) = -16 + 2k^2 + 6h^2 + k^2h + h^3$

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③ Problem: Expand e^y to nth order about $y=2$

Solution: $e^y = e^{2+(y-2)} = e^2 \cdot e^{y-2}$

for u near 0, $e^u \approx 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$

so $e^y \approx e^2 \cdot \left(1 + (y-2) + \frac{(y-2)^2}{2} + \frac{(y-2)^3}{6} + \cdots + \frac{(y-2)^n}{n!}\right)$

$\approx e^2 + e^2(y-2) + \frac{e^2}{2}(y-2)^2 + \cdots + \frac{e^2}{n!}(y-2)^n.$

Isn't it also true that

$e^y \approx 1 + y + \frac{y^2}{2} + \cdots + \frac{y^n}{n!}$ near 0, not near 2

Ex: $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{n=0}^{\infty} \frac{(2+(y-2))^n}{n!}$

(MATH 121-level

④ Q: what is a "saddle point"?

A: A saddle point is a critical point near which ~~...~~ the function takes values both larger and smaller than at the point itself

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⑤ Q: Expand $f(x) = \dfrac{e^{3x^2}}{1+5x^3}$ to 6th order about $x=0$

A: Note $f(x) = e^{3x^2}\left(\dfrac{1}{1+5x^3}\right)$

Know: $e^u \approx 1+u+\dfrac{u^2}{2}+\dfrac{u^3}{6}+\dfrac{u^4}{24}+\dots$

$\dfrac{1}{1-v} \approx 1+v+v^2+v^3+v^4+\dots$ used $u=3x^2$

A: $x\to 0$, $3x^2\to 0$, so $e^{3x^2} \approx 1+(3x^2)+\frac{1}{2}(3x^2)^2+\frac{1}{6}(3x^2)^3$

use $v=-5x^3$ correct to 6th order

stop here for 6th order

As $x\to 0$, $5x^3\to 0$, so $\dfrac{1}{1+5x^3} \approx 1+(-5x^3)+(-5x^3)^2$

correct to 6th order

so, to 6^{th} order,

$$f(x) \approx (1-5x^3+25x^6)\left(1+3x^2+\frac{9}{2}x^4+\frac{9}{2}x^6\right)$$

$$\approx 1+3x^2-5x^3+\frac{9}{2}x^4-15x^5+29\frac{1}{2}x^6.$$

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⑥ Q: $\frac{d}{dx}\left(\frac{e^{3x^2}}{1+5x^3}\right) = \frac{\frac{d}{dx}(e^{3x^2})(1+5x^3) - e^{3x^2}\cdot\frac{d}{dx}(1+5x^3)}{(1+5x^3)^2}$

$$= \frac{e^{3x^2}\left(\frac{d}{dx}3x^2\right)(1+5x^3) - e^{3x^2}\,15x^2}{(1+5x^3)^2},$$

$$= \frac{e^{3x^2}}{(1+5x^3)^2}\left(6x\cdot(1+5x^3)-15x^2\right)$$

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⑦ Expand $\frac{5}{3+2x}$ about $x=0$

observe: $\frac{5}{3+2x} = \frac{5}{3}\cdot\frac{1}{1+\frac{2}{3}x} = \frac{5}{3}\cdot\frac{1}{1-(-\frac{2}{3}x)}$

$$\approx \frac{5}{3}\left(1+\left(-\frac{2}{3}x\right)+\left(-\frac{2x}{3}\right)^2+\left(-\frac{2}{3}x\right)^3+\dots\right)$$

$$= \frac{5}{3}-\frac{10}{9}x+\frac{20}{27}x^2-\frac{40}{81}x^3+\dots$$

About $x=2$, $\frac{5}{3+2x} = \frac{5}{7+2(x-2)} = \frac{5}{7}\,\frac{1}{1-(-\frac{2}{7}(x-2))}$

⑧ Newton's law of cooling?

Don't have to memorize

Fact: To solve equation $y' = k(y + b)$

switch to variable $z = y + b$

Then $z' = kz$ so $z = C \cdot e^{kt}$.

need to know

(so $y = z - b = Ce^{kt} - b$)

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⑨ Expand $e^{\frac{1}{1-x}}$ to 3rd order about $x = 0$

Solution: Know $\frac{1}{1-x} \approx 1 + x + x^2 + x^3$ to 3rd order

$e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{6}$ to 3rd order

so

$$e^{\frac{1}{1-x}} \approx e^{1+x+x^2+x^3} \approx e \cdot e^{x+x^2+x^3}$$

(expand e^z about $z=1$)

$$\approx e\left(1 + (x+x^2+x^3) + \frac{1}{2}(x+x^2+x^3)^2 + \frac{1}{6}(x+x^2+x^3)^3\right) \approx$$

$$e\left(1 + x + \left(1+\frac{1}{2}\right)x^2 + \left(1 + \frac{1}{2}\cdot 2 + \frac{1}{6}\right)x^3 + \dots\right) \approx e + ex + \frac{3e}{2}x^2 + \frac{13e}{6}$$

(10) Asymptotics at $x \to \infty$ of $\sqrt{x^4+3x^3} - x^2$?

Notice $x^4+3x^3 \sim x^4$ as $x \to \infty$

so $\sqrt{x^4+3x^3} \sim x^2$

$\Rightarrow$ problem is about cancellation between $\sqrt{x^4+3x^3}$, x^2

So: $\sqrt{x^4+3x^3} - x^2 = x^2\sqrt{1+\frac{3}{x}} - x^2 = x^2\left(\sqrt{1+\frac{3}{x}} - 1\right)$

extracted overall x^2

let $u = \frac{3}{x}$

(study $\sqrt{1+u} - 1$)

Expand $\underset{g(u)}{\sqrt{1+u}}$ about $u=0$; $g'(u) = \frac{1}{2\sqrt{1+u}}$

so $g(0) = 1$, $g'(0) = \frac{1}{2}$, $g(u) \approx 1 + \frac{1}{2}u +$ small

$\Rightarrow \sqrt{1+\frac{3}{x}} - 1 \approx 1 + \frac{1}{2}\frac{3}{x} + \left(\text{higher order in } \frac{1}{x}\right) - 1$

$\approx \frac{3}{2x}$

so $\sqrt{x^4+3x^3} - x^2 \sim x^2\left(\frac{3}{2x}\right) \sim \frac{3}{2}x$

Or:

$\sqrt{x^4+3x^2} - x^2 = \left(x^2 \cdot \frac{3}{x}\right) \cdot \underbrace{\frac{\sqrt{1+\frac{3}{x}} - 1}{(3/x)}} \sim 3x \cdot \frac{1}{2}$

$\frac{\sqrt{1+u} - 1}{u} \xrightarrow[u \to 0]{} (\sqrt{1+u})'\big|_{u=0} = \frac{1}{2}$