1A3

6. Applications of the Chain rule (11/10/2023)

Goals.

(1) Implicit differentiation
(2) Inverse trig functions
(3) Related rates

Last Time. Chain rule

Newton notation: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Leibnitz notation: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Application: suppose $y = \log x$ can rewrite as $x = e^y$

diff. both sides along curve: $1 = \frac{dx}{dx} = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \cdot \frac{dy}{dx}$

chain rule

$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ i.e. $\boxed{\frac{d(\log x)}{dx} = \frac{1}{x}}$

from chain rule: $\boxed{f' = f \cdot (\log f)'}$

Log. diff. rule

Implicit differentiation

Example: $4x^2+y^2=36$ defines a curve.
Find the line tangent to the curve at $(2, 2\sqrt{5})$

Interpret the relation as an equality of functions

$x \mapsto 4x^2+(y(x))^2$ and $x \mapsto 36$

Then along the curve $\frac{d}{dx}(4x^2+y^2)=\frac{d}{dx}(36)$

so

linear eqn for $\frac{dy}{dx}$ $\rightarrow$ $8x+2y\cdot\frac{dy}{dx}=0$ ← ok with answer

chain rule: $\frac{d(y^2)}{dx}=\frac{d(y^2)}{dy}\cdot\frac{dy}{dx}=2y\frac{dy}{dx}$

so $\frac{dy}{dx}=-\frac{8x}{2y}=-\frac{4x}{y}$ ← ok with answer

so at point $(2, 2\sqrt{5})$ slope is $-\frac{8}{2\sqrt{5}}=-\frac{4}{\sqrt{5}}$

so the tangent line is $\boxed{y=-\frac{4}{\sqrt{5}}(x-2)+2\sqrt{5}}$

2. Implicit Differentiation

(3) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.

$2y\frac{dy}{dx} = 12x^2 + 2$ so $\frac{dy}{dx} = \frac{6x^2+1}{y}$

so $\left.\frac{dy}{dx}\right|_{(2,6)} = \frac{25}{6}$, line is $\boxed{y = \frac{25}{6}(x-2) + 6}$

(4) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

Example: sanity-checks

write $y^2 = 4x^3 + ax$

$[a] = [x]^2$

$[y] = [x]^{3/2}$

if x has units, ax, x^3, y^2 all have units of x^3

diff get $\frac{dy}{dx} = \frac{12x^2 + a}{2y}$

(1) x^2, a both have same units - can be added

(2) $\left[\frac{12x^2+a}{2y}\right] = \frac{[x]^2}{[x]^{3/2}} = [x]^{1/2}$

$\left[\frac{dy}{dx}\right] = \frac{[y]}{[x]} = \frac{[x]^{3/2}}{[x]^{1}} = [x]^{1/2}$ ✓

units mismatch

suppose we wrote $\frac{dy}{dx} = \frac{12x + a}{2y}$

or $\frac{dy}{dx} = \frac{12x^2 + a}{y^2}$

Inverse trig fcns

Example of inverse function: the square root.

direct function $f(x) = x^2$

The inverse $g(y)$ answers question "which x value solves $y = x^2$"

Issue ① equation $y = f(x)$ need not have any solutions

⇒ **domain of inverse = image of f**

(here need $y \geq 0$)

② equation $y = f(x)$ can have multiple solutions

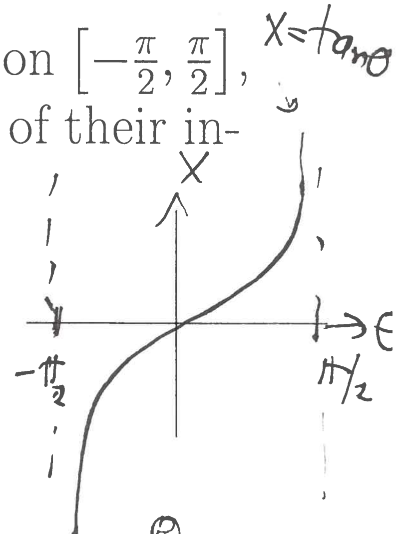
⇒ **restrict domain of f so only one x value qualifies**

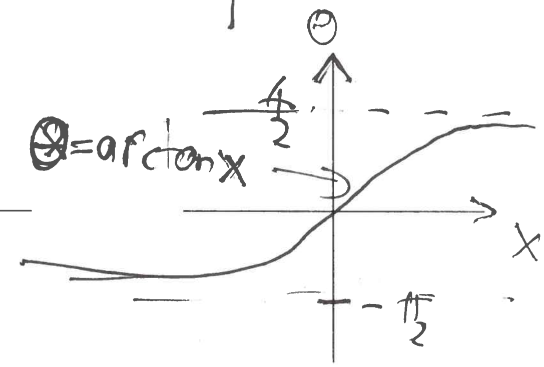
eg $\sqrt{y}$ is defined to be the non-negative solution to $x^2 = y$.

eg $f(x) = x^2$ on $[0, \infty)$ has an inverse
$f(x) = x^2$ on $\mathbb{R}$ doesn't.

3. Inverse trig functions

(7) Draw on the following axes graphs of $\sin\theta$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta$ on $[0, \pi]$ and $\tan\theta$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then of their inverse functions





range of $\sin\theta$ is $[-1,1]$ choose $\arcsin x = \theta$ if $\sin\theta = x$ and $\theta \in [-\pi/2, \pi/2]$ (defined if $-1 \le x \le 1$)

for $x \in [-1,1]$ $\arccos x = \theta$ if $\cos\theta = x$ and $0 \le \theta \le \pi$.

for all $x \in (-\infty, \infty)$ $\arctan x = \theta$ if $\tan\theta = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(8) Evaluation

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$ and $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (because $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$)

$\frac{31\pi}{11}$ solves $\sin\theta = \sin\left(\frac{31\pi}{11}\right)$

but $\frac{31\pi}{11} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ instead, $\sin(\theta + 2\pi) = \sin\theta$

$$\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9}{11}\pi\right) =$$

$$= \sin\left(\pi - \frac{9}{11}\pi\right) = \sin\left(\frac{2}{11}\pi\right)$$

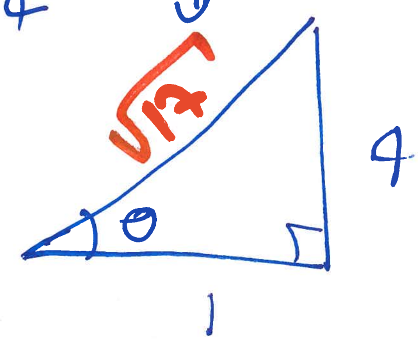
$\sin\theta = \sin(\pi - \theta)$

so answer is $\boxed{\frac{2}{11}\pi}$

(b) (Final 2015) Simplify $\sin(\arctan 4)$

idea: draw triangle with given tan, use Pythagor.

let $\theta = \arctan 4$



so $\sin\theta = \frac{4}{\sqrt{17}}$

(c) Find $\tan(\arccos(0.4))$

$$\boxed{\begin{aligned} \frac{d(\arcsin x)}{dx} &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d(\arccos x)}{dx} &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d(\arctan x)}{dx} &= \frac{1}{1+x^2} \end{aligned}}$$

Aside: on $y = f(x)$ & $x = f^{-1}(y)$

Note: $\arcsin x$ takes $\cdot x \in [-1, 1]$ gives <u>angle</u> θ

$x = \frac{1}{\sin\theta}$ takes angle θ gives $|x| > 1$

don't write $\sin^{-1} x$