Math 100A 1A1, lecture 14

Last time: **Multivariable optimization**

① On closed, bounded domain $f(x,y,..)$ has max & min

② Max/min occur at one of (i) critical pts $\left(\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \dots = 0\right)$

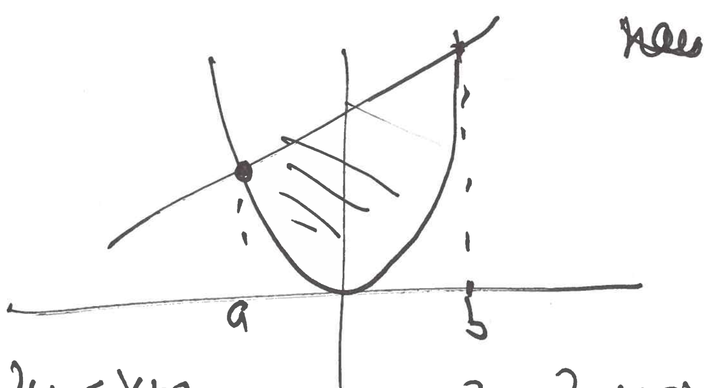
(ii) singular pts

(iii) boundary

③ on boundary need to optimize segment-by-segment using 1d techniques if $f(x,y)$, 2d techniques if $f(x,y,z)$,

④ If domain is unbounded need to handle behaviour "at infinity" (= asymptotically) to address point ①.

Question: domain is bdd by $y = x+2$, $y = x^2$



boundary is $\{y = x+2,\ a \le x \le b\} \cup \{y = x^2 :\ a \le x \le b\}$

need to study traces of f over each segment.

$x \mapsto f(x, x+2)$, $x \mapsto f(x, x^2)$ on $[a,b]$

Problem: find critical points of $f(x, y) = x^2y + y^3 - 12y$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy \\ \dfrac{\partial f}{\partial y} = x^2 + 3y^2 - 12 \end{cases}$$

So need to solve system $\begin{cases} 2xy = 0 \\ x^2 + 3y^2 = 12 \end{cases}$

From 1st equation we have either $x = 0$ or $y = 0$

(1) if $x = 0$, 2nd equation reads $3y^2 = 12$ so $y = \pm 2$

get solutions $(0, 2, -16)$, $(0, -2, 16)$

(2) if $y = 0$, 2nd equation reads $x^2 = 12$, so $x = \pm 2\sqrt{3}$.

get points $(\pm 2\sqrt{3}, 0, 0)$

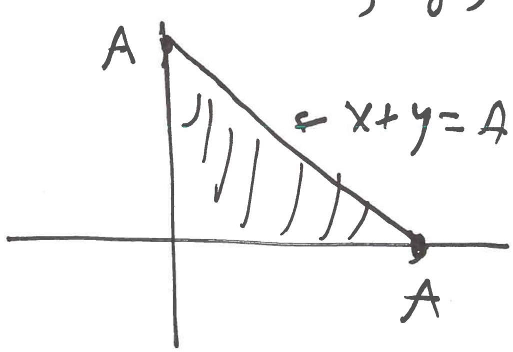
Problem: Given $A > 0$, find $x, y, z \geq 0$ so that $x + y + z = A$ and xyz largest.

Since $z = A - x - y$, we want to optimise

$$f(x, y) = xy(A - x - y)$$

on domain $x \geq 0$, $y \geq 0$, $A - x - y \geq 0 \Leftrightarrow x + y \leq A$

(1)



② In the interior, $\frac{\partial f}{\partial x} = y(A-x-y) - xy$

$$= Ay - 2xy - y^2$$

$$\Rightarrow \frac{\partial f}{\partial y} = Ax - 2xy - x^2$$

Critical pts at

$$\begin{cases} Ay - 2xy - y^2 = 0 \\ Ax - 2xy - x^2 = 0 \end{cases}$$

in the interior of our domain $x>0$, $y>0$. Dividing 1st equation by y, 2nd by x get

$$\begin{cases} A = 2x + y \\ A = 2y + x \end{cases}$$

subtracting equations gives $x - y = 0$ so $x = y = \frac{A}{3}$

③ On boundary either $x=0$, $y=0$, or $z = A - x - y = 0$

so $f(x,y) = 0$

At $x = y = z = \frac{A}{3}$, $f(\frac{A}{3}, \frac{A}{3}) = (\frac{A}{3})^3 > 0.$

so max is $(\frac{A}{3})^3$, attained at $(\frac{A}{3}, \frac{A}{3})$.

Aside: we proved $xyz \le \left(\frac{x+y+z}{3}\right)^3$

$\iff$ $(xyz)^{1/3} \le \frac{x+y+z}{3}$ for $x,y,z \geq 0$

"inequality of the means"

equality only if $x=y=z$.

(in general $\left(\prod_{i=1}^{n} x_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n} x_i$)

Variant: Also $\frac{x+y+z}{3} \le \sqrt{\frac{x^2+y^2+z^2}{3}}$

Q: Have $k(T) = T(40-T) + 10T$

is the function increasing/decreasing near $T=20$?

use the linear approximation.

Solution 1: $k'(T) = 40 - T - T + 10 = 50 - 2T$

so $k'(20) = 10$, $k(20+h) \approx \underset{\substack{\uparrow \\ k(20)}}{600} + \underset{\substack{\uparrow \\ k'(20)}}{10}h$

Solution 2: (by hand)

$k(20+h) = (20+h)(20-h) + 10(20+h) = 400 - h^2 + 200 + 10h$

$= 600 + 10h - h^2 \approx 600 + 10h$ to 1st order in h.

Σ notation

$\sum_{k=a}^{b} f(k)$ means "add the values $f(a), f(a+1), f(a+2), \ldots, f(b)$"

instead of $1+x+x^2+\cdots+x^7$

write $\sum_{k=0}^{7} x^k$

what about $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -$

this is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

Final 2016, Q13b, lab <u>not</u> examinable this year

(Not 2023 material)

Let $T_n(x)$ = nth degree Taylor expansion of $f(x) = \log x$ about $x=1$. ~~For~~ For which n is $T_n(1.1)$ an over/under estimate?

<u>Solution:</u> By Lagrange remainder formula,

$$f(x) = T_n(x) + R_n(x), \quad R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1}$$

where ξ <u>between</u> x, a.

Here, $\log(1.1) - T_n(1.1) = \frac{1}{(n+1)!} \cdot f^{(n+1)}(\xi) \cdot (0.1)^{n+1}$

with $1 < \xi < 1.1$.

Now $(\log x)' = \frac{1}{x}$, $f^{(2)}(x) = -\frac{1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$, $f^{(4)} = -\frac{6}{x^4}$, ...

since $x > 0$, we see that $f^{(n)}(x) > 0$ if n odd
< 0 if n even

$\Rightarrow R_n(1.1) > 0$ if $n+1$ odd $\Leftrightarrow$ n even
< 0 if $n+1$ even, $\Leftrightarrow$ n odd

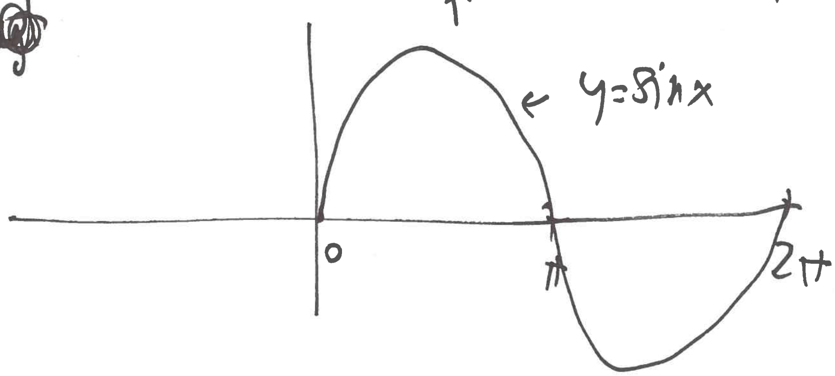
Say $|f(x) - \sin x| \le \frac{1}{3}$ for all x, f, f', f'' all cts.

Q: Show $f''(c) = 0$ for some $c \in (0, 2\pi)$

If $f'' \neq 0$ on $(0, 2\pi)$ then either f'' always > 0
or always < 0

there. So f is either concave up or concave down on $[0, 2\pi]$ so f crosses x-axis at most twice.

f' " " " " once;



f' either increasing or decreasing.

On $[0, \frac{\pi}{2}]$ $\quad f(0) \leq \frac{1}{3}$ (within $\frac{1}{3}$ of $\sin 0$)

$f(\frac{\pi}{2}) \geq \frac{2}{3}$ (" " " $\sin \frac{\pi}{2}$)

so f must increase somewhere between $[0, \frac{\pi}{2}]$

$f' > 0$ <u>somewhere</u> on $[0, \frac{\pi}{2}]$

on $[\frac{\pi}{2}, \frac{3\pi}{2}]$, $f(\frac{\pi}{2}) \geq \frac{2}{3}$, $f(\frac{3\pi}{2}) \leq -2/3$

so f must decrease somewhere, $f' < 0$ somewhere

on $(\frac{\pi}{2}, \frac{3\pi}{2})$

so $f'' < 0$ on $(0, 2\pi)$, so $f' < 0$ on $(\frac{3\pi}{2}, 2\pi)$ also

f' is decreasing, so

But $f(2\pi) \geq \frac{2}{3} > f(\frac{3\pi}{2})$ impossible

Problem

$\ddot{y} = -g + k(\dot{y})^2 \qquad \dot{y} = \frac{dy}{dt}$

(a) find DE satisfied by $v(t) = \frac{dy}{dt}$

Solution:

$$\dot{v} = -g + k v^2$$

ors $\dot{v} = \ddot{y} = = -g + k(\dot{y})^2 = -g + kv^2$

(b) find steady state

want $v(t) \equiv v_0$ to be a solution

that is $0 = -g + k v_0^2$ so $v_0 = \sqrt{g/k}$

(c) set $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$

check $(\cosh x)' = \sinh x \qquad (\tanh x)' = 1 - (\tanh x)^2$

$(\sinh x)' = \cosh x \qquad \tanh x = \frac{\sinh x}{\cosh x}$

eg $\left(\frac{e^x + e^{-x}}{2}\right)' = \frac{1}{2}(e^x - e^{-x})$

(d) try $v(t) = A \tanh(\alpha(t - t_0))$

then $\dot{v} = A\alpha \cdot (1 - \tanh^2(\alpha(t - t_0)))$

$= A\alpha - \frac{\alpha}{A} \cdot (A \tanh(\alpha(t - t_0)))^2$

so for this Ansatz,

$$\dot{V} = A\alpha - \frac{\alpha}{A} V^2$$

it solves $\dot{V} = -g + kV^2$

if $\begin{cases} A\alpha = -g \\ -\frac{\alpha}{A} = k \end{cases}$ $\Rightarrow -\alpha^2 = -gk$

so $\alpha = \sqrt{gk}$

$$A = -\frac{g}{\alpha} = -\sqrt{\frac{g}{k}} = V_0$$

so solution is

$$\boxed{V(t) = -\sqrt{\frac{g}{k}} \tanh\left(\sqrt{gk}\,(t - t_0)\right)}$$

Problems $x = y\sin(x+y)$

① $\dfrac{dx}{dy} = ?$

diff both sides, $\dfrac{dx}{dy} = \sin(x+y) + y\cos(x+y)\cdot\left(\dfrac{dx}{dy} + 1\right)$

so $\left(1 - y\cos(x+y)\right)\dfrac{dx}{dy} = \sin(x+y) + y\cos(x+y)$

so $$\boxed{\dfrac{dx}{dy} = \dfrac{\sin(x+y) + y\cos(x+y)}{1 - y\cos(x+y)}}$$

② Evaluate at $(0, \pi)$

at $(0,\pi)$, $x+y = \pi$, $\sin\pi = 0$, $\cos\pi = -1$

$$\left.\dfrac{dx}{dy}\right|_{(0,\pi)} = \dfrac{0 + \pi(-1)}{1 - \pi(-1)} = -\dfrac{\pi}{1+\pi}$$

③ approximate $x(3)$ along the curve.

Linear approx: $x(3) \approx x(\pi) + x'(\pi)\cdot(3-\pi)$

$$= 0 - \dfrac{\pi}{1+\pi}\cdot(3-\pi)$$

so point is about $\left(\dfrac{\pi(\pi-3)}{1+\pi}, 3\right)$