1A1

7. Curve sketching; ~~Taylor expansion~~ (20/10/2023)

Goals.

(1) Convexity
(2) Curve sketching

Last Time. **Implicit diff.** If $f(x,y) = g(x,y)$ along some curve, can diff this relation wrt x + solve for y' as fcn of x, y

Advice: use Leibnitz notation. Eg. $\frac{d(\log y)}{dy} = \frac{1}{y}$

but $\frac{d(\log y)}{dx} = \frac{1}{y}\frac{dy}{dx}$

Related rates: Can also have x, y depending on variable s. Then can diff relation wrt s, get relation between $x, y, \frac{dx}{ds}, \frac{dy}{ds}$.

Inverse trig: (1) defined arcsin, arccos, arctan (needed to restrict domain of sin, cos, tan)

(2) differentiated: memorize derivatives.

Math 100A – WORKSHEET 7
CURVE SKETCHING

1. Convexity and Concavity

(1) Consider the curve $y = x^3 - x$.

(a) ⋆ Find the line tangent to the curve at $x = 1$.

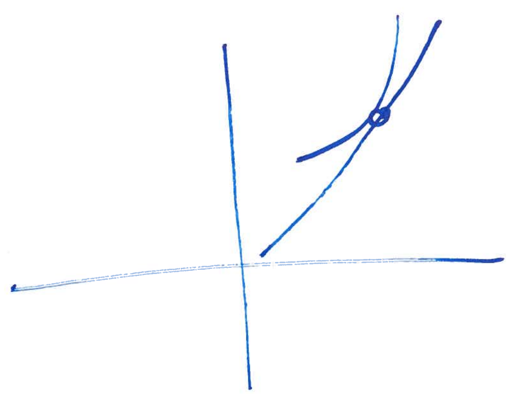
$\frac{dy}{dx}\Big|_{x=1} = \left[3x^2-1\right]_{x=1} = 2$, point: $(1,0)$ line:

$y = 2(x-1)$

(b) ⋆⋆ Near $x = 1$, is the line above or below the curve? Hint: how does the slope of the curve behave to the right and left of the point?

slope $3x^2-1$ is increasing near $x=1$

so:



If $x>1$, slope $y'>2$ curve moves above line

If $x<1$, slope $y'<2$ curve moves above line

Or: $x^3-x-2(x-1) = (x-1)(x^2+x-2) = (x-1)^2(x+2)$

Date: 18/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

so $x^3-x = 2(x-1) + 3(x-1)^2 + (x-1)^3 > 2(x-1)$ if x close

since $(x-1)^2 >> (x-1)^3$ as $x \to 1$.

Conclusion: $f'' > 0 \iff f'$ increasing $\iff$ tangent lines under the graph

$f'' < 0 \iff f'$ decreasing $\iff$ tangent lines above the graph

Def: In first ~~case~~ case say f is concave up

In second case " " " " down.

If f is cts at $x=a$, concavity changes at $x=a$

say that $x=a$ is an inflection point.

(here $f''(a)=0$ or undef)

(But $f(x)=x^4$ has $f''(0)=0$ but $f''(x)=12x^2$

does not change sign there, so no inflection)

(2) For each curve find its domain; where is it concave up or down? Where are the inflection points.

(a) $y = x \log x - \frac{1}{2}x^2$.

$y' = \log x + x \frac{1}{x} - x = \log x + 1 - x$, $y'' = \frac{1}{x} - 1$

Domain: $x > 0$ (domain of $\log x$)

concave up if $\frac{1}{x} - 1 > 0 \Leftrightarrow x < 1$ i.e. on $(0,1)$

not $(-\infty, 1)$ (fn undefined if $x \leq 0$)

concave down if .. $x > 1$, on $(1, \infty)$

$\Rightarrow$ inflection point at $x = 1$ (or at $(1, -\frac{1}{2})$)

(a) Consider the curve $y = \sqrt[3]{x}$. Where is it continuous? Find where it is concave up and down.

domain $(-\infty, \infty)$ $y'' = -\frac{2}{9} x^{-5/3} = -\frac{2}{9} \frac{1}{(\sqrt[3]{x})^5}$

if $x < 0$, $y'' > 0$, if $x > 0$, $y'' < 0$

so concave up on $(-\infty, 0)$, down on $(0, \infty)$

inflection pt at $(0, 0)$.

$y(0) = 0$, but $y'(0)$ undef
$y''(0)$ "

singular point

(c) What are the intervals of concavity? Any inflection points?

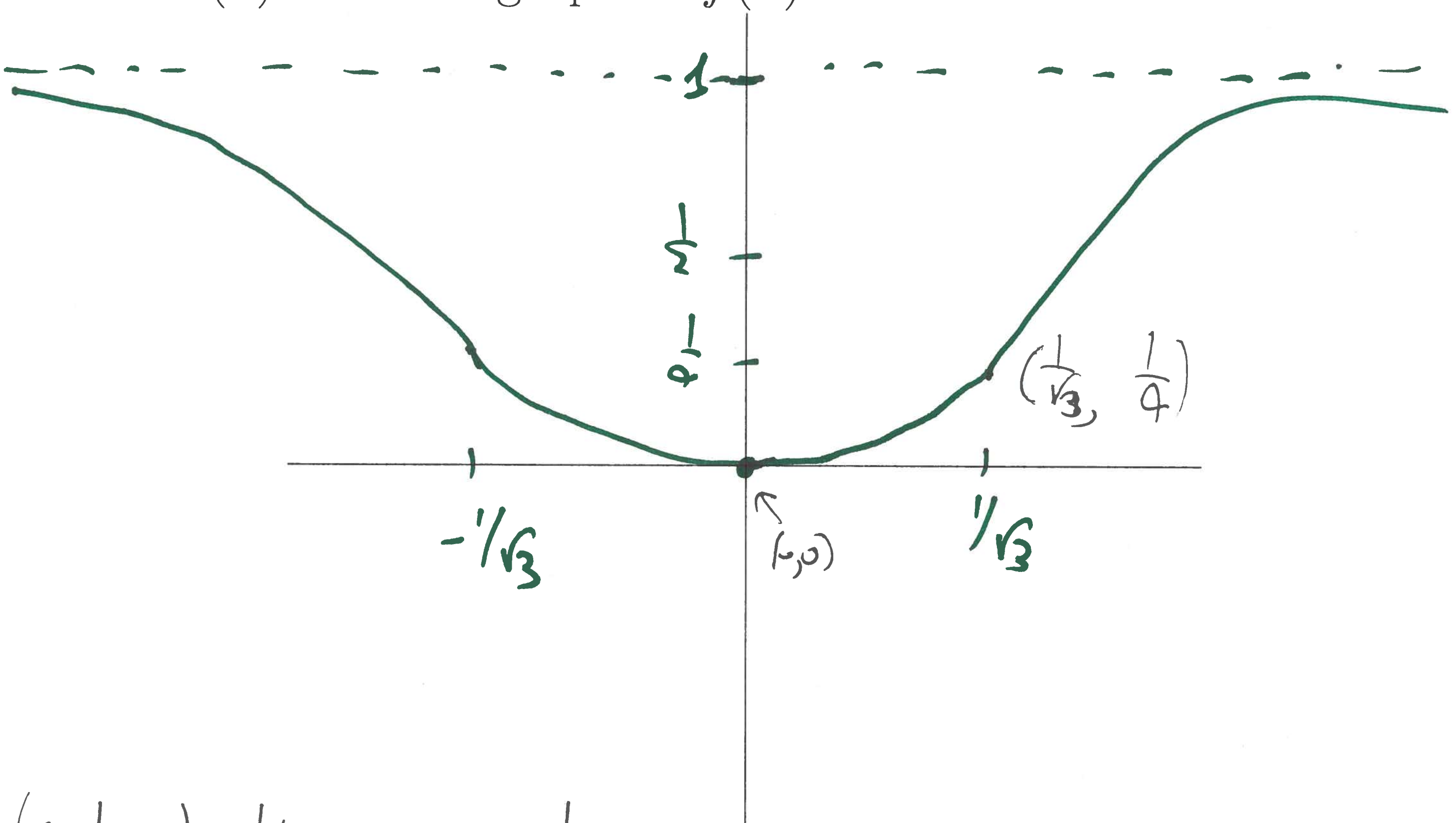
Since $\frac{2}{(x^2+1)^3} > 0$ always, $f''(x)$ has same sign as $1-3x^2$.

So f is concave down on $(-\infty, -\frac{1}{\sqrt{3}})$, and on $(\frac{1}{\sqrt{3}}, \infty)$,

" up on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Has inflection pts at $x = \pm\frac{1}{\sqrt{3}}$, $(\pm\frac{1}{\sqrt{3}}, \frac{1/3}{1+1/3}) = (\pm\frac{1}{\sqrt{3}}, \frac{1}{4}$

(d) Sketch a graph of $f(x)$.



(only believe aspects we put in deliberately)

(4) ⋆⋆ Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotices there?

domain is $(-\infty, \infty)$, $f(x) > 0$ for all x, $f(0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}}$

as $x \to \pm\infty$, $-\frac{(x-\mu)^2}{2\sigma^2} \sim -\frac{x^2}{2\sigma^2}$ so $f(x)$ will decay rapidl to 0

so have horiz. asymptote $y=0$

(b) What are the intervals of increase/decrease? The local and global extrema?

$f'(x) = -\frac{(x-\mu)}{\sqrt{2\pi\sigma^2}\,\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ has same sign as $-(x-\mu)$

so f is increasing when $x < \mu$

f " decreasing where $x > \mu$

Have a critical pt at $\left(\mu, \frac{1}{\sqrt{2\pi\sigma^2}}\right)$, is global maximum

2. Curve sketching

(3) ⋆⋆ Let $f(x) = \frac{x^2}{x^2+1}$ for which $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotices there?

f defined on $(-\infty, \infty)$ (since $x^2+1 > 0$ for all x).

$f(0) = 0$ & if $f(x) = 0$ then $x^2 = 0$ so $x = 0$

As $x \to \pm\infty$, $\frac{x^2}{x^2+1} \sim \frac{x^2}{x^2} = 1$, Horiz asymptote $y=1$ as $x \to \infty$, as $x \to -\infty$

no vertical asymptote

(b) What are the intervals of increase/decrease? The local and global extrema?

Since $\frac{2}{(x^2+1)^2} > 0$ always, $f'(x) > 0$ when $x > 0$
$f'(x) < 0$ " $x < 0$

so f increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$
critical point $(0,0)$ at $x=0$, is a minimum

(c) What are the intervals of concavity? Any inflection points?

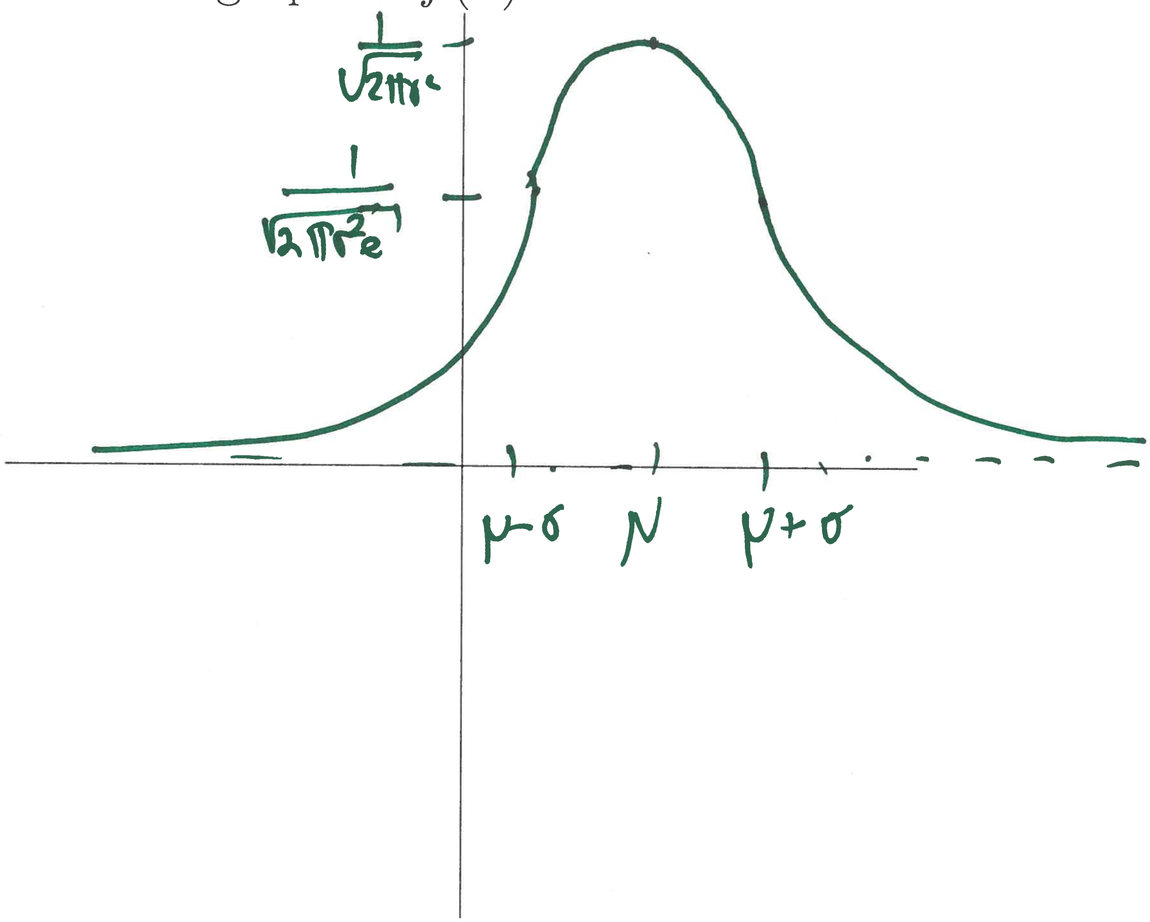
$f''(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \cdot \left(\frac{(x-\mu)^2}{\sigma^2} - 1\right)$

So $f'' > 0$ if $(x-\mu)^2 > \sigma^2$ on $(-\infty, \mu-\sigma)$, on $(\mu+\sigma, \infty)$

$f'' < 0$ if $(x-\mu)^2 < \sigma^2$, on $(\mu-\sigma, \mu+\sigma)$

Have inflection pts at $\left(\mu \pm \sigma, \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}}\right)$

(d) Sketch a graph of $f(x)$.



(5) (Final, December 2007) ⋆⋆ Let $f(x) = x\sqrt{3-x}$.

(a) Find its domain, intercepts, and asymptotics at the endpoints.

(b) What are the intervals of increase/decrease? The local and global extrema?

$$f'(x) = \sqrt{3-x} - \frac{x}{2\sqrt{3-x}}$$ ← not pdt

$$= \frac{2(3-x)-x}{2\sqrt{3-x}} = \frac{6x-2}{2\sqrt{3-x}} = \frac{3x-1}{\sqrt{3-x}}$$ ← is pd