Math 535, lecture 37, 28/4/23

Iwasawa Decomposition

Previously: Cartan involutions + Polar decomposition

Setup: G ss. ctd Lie gp, $\mathfrak{g}$ = Lie G; $\Theta \in \mathrm{Aut}(G)$
(global) Cartan involution: $\Theta^2 = id_G$

(1) $\theta = d\Theta \in \mathrm{Aut}(\mathfrak{g})$ has $B_\theta(X,Y) = B(X, \theta Y)$
is neg. def.

(2) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ± 1 eigenspaces for θ.
then
$G^\Theta = K$ = analytic subgp with Lie alg $\mathfrak{k}$.
with $K \supset Z = Z(G)$, K/Z cpt.
(typically Z is finite, K cpt)

(3) Map $K \times \mathfrak{p} \to G$ $(k, X) \mapsto k\exp X$ is a diffeo
($\Rightarrow K$ is a deformation retract of G)

(4) If Z is finite, K is a max'l cpt subgp
(e.g. G/K symmetric space, cpt subgps fix pts)

Can choose basis in $\mathfrak{g}$ s.t. $Ad(G) = Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ is closed under $g \mapsto {}^t g^{-1}$, $\Theta(g)$ = restriction of this to $Ad(G)$

$\theta(X) = -X^t$.

Then $\mathfrak{k}$ = antisymmetric matrices $\quad K = Ad(G) \cap O(\mathfrak{g})$

$\mathfrak{p}$ = symmetric matrices

$G \cong K \times \exp \mathfrak{p}$ is usual polar decomposition.

In $GL_n(\mathbb{R})$ also have Gram-Schmidt process: can write any $g \in GL_n(\mathbb{R})$ uniquely as $g = nak$ where $n = \begin{pmatrix} 1 & * \\ & \ddots \\ & & 1 \end{pmatrix}$, $a = \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_n} \end{pmatrix}$, $k \in O(n)$.

<u>Thm:</u> Similarly can "line up" co-ords so that $Ad(G) = NAK$, K as above

A: diagonal matrices in G

N = upper triag. unipotents in G.

<u>Motivation 1:</u> Can realize rep'n theory of G by inducing rep'ns

$$\mathrm{Ind}_{NAM}^{G} (1 \otimes \chi \otimes \sigma)$$

where $\chi \in Hom(A, \mathbb{C}^*)$, $\sigma \in \hat{M}$, $M = Z_K(A)$

can realise the induced rep'n on space of fcn on K/M.

Motivation 2: NA is a co-ord system on symmetric space G/K

Example: $\mathbb{H}^{(n+1)} = \left\{ \underline{x} + iy \;\middle|\; \begin{matrix} \underline{x} \in \mathbb{R}^n \\ y \in \mathbb{R}_{>0} \end{matrix} \right\}$ with metric

$$\frac{d\underline{x}^2 + dy^2}{y^2}$$

$$\mathrm{Isom}(\mathbb{H}^{(n+1)}) \supset N = \{ n(\underline{x}') \} \quad n(\underline{x}')(\underline{x} + iy) = (\underline{x}' + \underline{x}) + iy$$

$$\supset A = \{ a(t) \} \quad a(t) \cdot (\underline{x} + iy) = e^t(\underline{x} + iy)$$

A normalises N, and $NA \cong \mathbb{H}^{(n+1)}$;

$$\underline{x} + iy = n(\underline{x})\, a(\log y) \cdot i$$

Fact: $\mathbb{H}^{(n+1)}$ is a symmetric space,

$$G = \mathrm{Isom}(\mathbb{H}^{(n+1)}) \cong SO(n,1)$$

(if $n=2$, $\cong PSL_2(\mathbb{R})$
$n=3$, $\cong PSL_2(\mathbb{C})$)

Pf: have bijection $C: \mathbb{H}^{(n+1)} \to B^{(n+1)}$ s.t. metric

is radial. $\Rightarrow K = \mathrm{Stab}_G(i) = O(n+1)$, including the reflection at origin.
so $\mathbb{H}^{(n+1)}$ is a symmetric space, $G = NAK$.

Definitions and thms

Fix $\mathfrak{a} \subset \mathfrak{p}$ max'l abelian, real cartan subalgebra
$r = \dim_{\mathbb{R}} \mathfrak{a}$ real rank.

$\mathrm{ad}\, \mathfrak{a} \subset \mathfrak{p} \Rightarrow$ symmetric wrt $B_\mathfrak{g}$ so
$\Rightarrow$ diagonable $/\mathbb{R}$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \qquad \text{for } \Sigma \subset \mathfrak{a}^* \setminus \{0\}$$

"restricted roots".

Lemma: $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{p}) \oplus (\mathfrak{g}_0 \cap \mathfrak{k}) = \mathfrak{a} \oplus \mathfrak{m}$

$\mathfrak{m} = \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$; $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ $\left(\begin{array}{l} \theta(H) = -H \\ \text{if } H \in \mathfrak{a} \end{array} \right)$

$\Downarrow$

$-\Sigma = \Sigma$

Let $\mathfrak{b} \subset \mathfrak{m}$ be a max'l torus, then $\mathfrak{h} := \mathfrak{a} + \mathfrak{b}$ is max'l abelian, so $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalg of $\mathfrak{g}_{\mathbb{C}}$

$\Rightarrow$

$$\Sigma = \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Phi(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}}) \} \setminus \{0\}$$

the roots are real on $\mathfrak{a} \oplus i\mathfrak{b}$

As with cpt grps choose notion of positivity for $\mathfrak{a}^*$ (fix basis of $\mathfrak{a}$, ...)

$\Rightarrow$ Σ^+ positive roots, $\Delta \subset \Sigma^+$ simple roots

$\Sigma^+ \subseteq \bigoplus_{\alpha \in \Delta} \mathbb{Z}_{\geq 0}\alpha$, set $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$

$$\bar{\mathfrak{n}} = \Theta\mathfrak{n} = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$$

clearly $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}$

if $X_\alpha \in \mathfrak{g}_\alpha$, $X_\alpha + \Theta X_\alpha \in \mathfrak{k}$

$\Theta X_\alpha \in \mathfrak{g}_{-\alpha}$, $X_\alpha - \Theta X_\alpha \in \mathfrak{p}$

$\Rightarrow$ $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$

(Goal: $G = N \times A \times K$)

<u>Def:</u> A: analytic subgp with Lie $A = \mathfrak{a}$
N = " " " Lie $N = \mathfrak{n}$

<u>Prop:</u> A, N closed subgps, $\exp : \mathfrak{a} \to A$
$\exp : \mathfrak{n} \to N$
are diffeos (A normalizes N)

<u>Pf:</u> As $H \to \infty$ in $\mathfrak{a}$, must have $|\alpha(H)| \to \infty$ for some root $\Rightarrow$ as $H \to \infty$, $Ad(\exp(H)) \to \infty$ in $Ad(G)$
so $\exp(H) \to \infty$ in G,
so $\exp_A$ (always a hom) is injective proper
$\Rightarrow$ diffeo, image closed since A, $\mathfrak{a}$ loc. cpt.

If we take H in positive Weyl chamber
then $\exp(tH)$ <u>uniformly expands</u> $\mathfrak{n}$, N
$\exp(-tH)$ uniformly contracts $\mathfrak{n}$, N.

$\exp_N : \mathfrak{n} \to N$ is locally diffeo $\Rightarrow$ globally so

can also prove $\overline{N} = N$ (N is closed)
(see notes)

Cor: NA is a closed subgp, $\exp: \mathfrak{n} \oplus \mathfrak{a} \to NA$ is a diffeo.

Lemma: G Lie gp, $S, T \subset G$ ctd subgps, s.t. $\operatorname{Lie} G = \operatorname{Lie} S \oplus \operatorname{Lie} T$. Then $m: S \times T \to G$ is everywhere regular, thus open.

Lemma: $S, T \subset G$ closed, T cpt then ST is closed

Thm: The map $N \times A \times K \to G$ is a diffeo

Pf: Know $N \times A \to NA$ is a diffeo.

$NA, K \subset G$ are closed, $\mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k} = \mathfrak{g}$

$\Rightarrow$ multiplication $NA \times K \to G$ is regular, open, has closed image. $\Rightarrow$ surjective.

Bijection iff $NA \cap K = \{1\}$ but eigen values of $\operatorname{Ad}(na)$ on $\mathfrak{g}$ are those of $\operatorname{Ad}(a)$ so if $na \in K$ then $\operatorname{Ad}(a) = 1$ (all ev. have $|\cdot| = 1$) so $a = 1$, $\operatorname{Ad}(n)$ is unipotent, $\operatorname{Ad}(k)$ ss so $n = 1$.