Math 535, lecture 36. 29/4/23

Setup: G ctd s.s. Lie gp, Lie alg. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

$\Theta \in \mathrm{Aut}(G)$ Cartan involution, diff $\theta \in \mathrm{Aut}(\mathfrak{g})$

s.t. $B_\theta(X,Y) = B(X, \theta Y)$ is neg. def.

Then

$\mathfrak{k}, \mathfrak{p}$ are $+1, -1$ eigenspaces

$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$

Previously: $K = \mathrm{Fix}(\Theta) \subset G$ closed subgp, ctd, contains $Z = Z(G)$, Lie $K = \mathfrak{k}$, K/Z cpt. Polar decomposition

$G \simeq K \times \exp \mathfrak{p}$. (diffeo)

Last time: $S = G/K$ is a symmetric space of non-positive curvature

If Z is finite, K is a max'l cpt subgp of G all are conjugate.

Today: roots & Weyl group for real Lie algebra $\mathfrak{g}$.

Recall: in adjoint rep'n, $\theta(X) = -X^t$, $\Theta(g) = {}^t g^{-1}$

$\mathfrak{k}$ = anti-symmetric matrices
$\mathfrak{p}$ = symmetric matrices
} wrt B_θ

Iwasawa Decomposition

Let $\mathfrak{a} \subset \mathfrak{p}$ be a max'l abelian subalgebra, a real Cartan subalgebra. ($\mathfrak{a}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ not a Cartan subalg.)

Since ad H are symmetric wrt B_θ for all $H \in \mathfrak{a}$, then are diagonalble.

Def: Call $r = \dim_{\mathbb{R}} \mathfrak{a}$ the real rank of G.

(Since $\mathfrak{a}$ is ad-diagonalble, $\mathfrak{a}_{\mathbb{C}}$ is contained in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$, so $r \leq \operatorname{rk} \mathfrak{g}_{\mathbb{C}}$)

(if G cpt, $\mathfrak{p} = \{0\}$ $r = 0$)

(Call G $\mathbb{R}$-split if $r = \operatorname{rk} \mathfrak{g}_{\mathbb{C}}$, ie if $\mathfrak{a}_{\mathbb{C}}$ is a Cartan subalg. of $\mathfrak{g}_{\mathbb{C}}$).

Lemma-def'n: $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{m}$ where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$

Pf: Since $\theta(\mathfrak{a}) = \mathfrak{a}$, θ acts on $Z_{\mathfrak{g}}(\mathfrak{a})$

so $Z_{\mathfrak{g}}(\mathfrak{a}) = Z_{\mathfrak{p}}(\mathfrak{a}) \oplus Z_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{a} \oplus Z_{\mathfrak{k}}(\mathfrak{a})$.

$\mathfrak{a}$ is max'l abelian.

Cor: Let $\mathfrak{b} \subset \mathfrak{m}$ be a max'l abelian subalg.
Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ is a Cartan subalg. of $\mathfrak{g}$.

Pf: $\mathfrak{b}$ is ad-diagonalizable since K is cpt

$$Z_{\mathfrak{g}}(\mathfrak{h}) = Z_{\mathfrak{a} \oplus \mathfrak{m}}(\mathfrak{b}) = \mathfrak{a} \oplus Z_{\mathfrak{m}}(\mathfrak{b}) = \mathfrak{a} \oplus \mathfrak{b}.$$

$\blacksquare$

(then $\mathfrak{b} + i\mathfrak{a} \subset \mathfrak{g}_{\mathbb{C}}$ is the max'l torus of the cpt form)

Def: The restricted roots $\Sigma = \Sigma(\mathfrak{g} : \mathfrak{a}) \subset \mathfrak{a}^*$, are the characters of $\mathfrak{a}$ occuring in the adjoint action on $\mathfrak{g}$ (other than 0)

Summary: $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ with

(1) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

(3) $\theta(H) = -H \Rightarrow$
$\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$
$\Rightarrow \Sigma = -\Sigma$

(4) $\Sigma = \{\alpha\restriction_{\mathfrak{a}} \mid \alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\}$, the roots ~~of~~ Φ are real on $\mathfrak{a} \oplus i\mathfrak{b}$.

<u>Example</u>: $G = SL_n(\mathbb{R})$, $\mathfrak{a} = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ $(\mathrm{tr} = 0)$
$\mathbb{R}$-split, so same roots as for $SL_n(\mathbb{C})$

<u>Example</u>: $G = SL_2(\mathbb{C})$ (thought of as a <u>real</u> group)

$\Theta(g) = {}^t\bar{g}^{-1}$ (transpose, complex conj, inverse)

$\Theta(X) = -{}^t\bar{X}$ $\quad \mathfrak{g} = \{X \in M_2(\mathbb{C}) \mid \mathrm{tr}\, X = 0\}$

$\mathfrak{k} = \mathfrak{su}(2) = \{X \mid \bar{X}^T = -X, \ \mathrm{tr}\, X = 0\}$

$\mathfrak{p} = \{X \mid \bar{X}^T = X, \ \mathrm{tr}\, X = 0\}$
$= \left\{ \begin{pmatrix} a & b \\ b^* & -a \end{pmatrix} \middle| \begin{matrix} a \in \mathbb{R} \\ b \in \mathbb{C} \end{matrix} \right\}$

$H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \mathfrak{p}$

$Z_{\mathfrak{p}}(H) = \mathbb{R} \cdot H$ $\qquad Z_{\mathfrak{g}}(H) = \mathbb{R} \cdot H + \mathbb{R} \cdot \begin{pmatrix} i & \\ & i \end{pmatrix}$
in $\mathfrak{k}$

$\mathfrak{m} = \mathfrak{b} = \mathbb{R} \cdot iH = \left\{ \begin{pmatrix} i\theta & \\ & -i\theta \end{pmatrix} \right\}$

(Max'l torus of $SL_2(\mathbb{C})$ is AM, $A = \left\{ \begin{pmatrix} e^{tH} & \\ & e^{-tH} \end{pmatrix} \right\}$

$M = \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \right\}$)

two restricted roots:

$\mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & z \\ & 0 \end{pmatrix} : z \in \mathbb{C} \right\}$, $\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & \\ z & 0 \end{pmatrix} \right\} z \in \mathbb{C} \}$

$\alpha(H) = 2$ $\qquad \dim_{\mathbb{R}} \mathfrak{g}_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_{-\alpha} = 2$

(in fact $\mathfrak{sl}_2\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$

$\mathfrak{a} \oplus \mathfrak{m}$ embedded in $\nearrow$: $\mathfrak{a}$ diagonally, $\mathfrak{m}$ via complex conj. in 2^{nd} copy.)

As before, choose a notion of positivity for $\mathfrak{a}^*$ (fix basis $\{H_i\}_{i=1}^r \subset \mathfrak{a}$. say $\alpha > \beta$ if first i s.t. $\alpha(H_i) \neq \beta(H_i)$ have $\alpha(H_i) > \beta(H_i)$) (can extend to basis of $\mathfrak{a} \oplus i\mathfrak{b}$ ensure that ordering is compatible with one on Φ)

Call $\alpha \in \Sigma^+$ simple if it's not a positive combo of positive roots, Δ = simple roots $\subset \Sigma^+$

As before, $\Sigma^+ \subseteq \bigoplus_{\alpha\in\Delta} \mathbb{Z}_{\geq 0}\alpha$, $n = \bigoplus_{\alpha\in\Sigma^+} g_\alpha$ is the subalg. generated by $\{g_\alpha\}_{\alpha\in\Delta}$.

pos roots

Set $\bar{n} = \Theta(n) = \bigoplus_{\alpha<0} g_\alpha$, so $g = \bar{n} \oplus \underbrace{a \oplus m}_{0} \oplus n$

nega roots

Prop: $g = n \oplus a \oplus k$

Pf: Fix bases $\{H_i\} \subset a$, $\{M_j\} \subset m$, $\{X_k\} \subset n$
Then
$$\{\Theta(X_k)\} \cup \{H_i\} \cup \{M_j\} \cup \{X_k\}$$
is a basis of g.

$\Rightarrow \{X_k - \Theta(X_k)\} \cup \{H_j\} \cup (\{M_j\} \cup \{X_k + \Theta(X_k)\})$
is also a basis.
now $X_k - \Theta(X_k), H_j \in p \mid M_j, X_k + \Theta(X_k) \subset k$

$g = p \oplus k$ so these are bases of the subspaces

$\Rightarrow$ in basis $\{X_k\} \cup \{H_i\} \cup (\{M_j\} \cup \{X_k + \Theta(X_k)\})$
we see
$$n \oplus a \oplus k = g.$$
$\blacksquare$

Let $A, N \subset G$ be the subgps corresp to $\mathfrak{a}, \mathfrak{n}$.
Let $M = Z_K(\mathfrak{a}) = Z_K(A)$.

<u>Goal:</u> $G \cong N \times A \times K$ (diffeo)

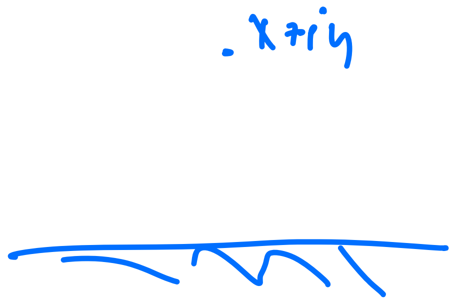
<u>Example:</u> $G = GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$
N = upper triangular unipotent $\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$,
A = diagonal, positive real entries
$K = O(n)$ or $U(n)$.

$G = NAK$ is Gram-Schmidt!

$M = \mathrm{diag}(\pm 1, \pm 1, \ldots, \pm 1)$ in $GL_n(\mathbb{R})$
$\mathrm{diag}(S^1, S^1, \ldots, S^1)$ in $GL_n(\mathbb{C})$

<u>Example:</u> $G = PSL_2(\mathbb{R})$ acting on its symmetric space $\mathbb{H} = \{x+iy \mid y>0\}$ by Möbius transform

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$



(metric is $\dfrac{dx^2+dy^2}{y^2}$)

clearly $z \mapsto z+b$ is an isometry
also $z \mapsto az$

action of $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{a} & \\ & 1/\sqrt{a} \end{pmatrix} \cdot z$

Iwasawa: NA acts simply transitively on $S = G/K$

here:
$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \cdot i = x + iy.$$

or $$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot i = x + iy$$

Similarly $\mathbb{H}^{(3)} = \left\{ \underline{x} + iy \,\middle|\, \begin{matrix} \underline{x} \in \mathbb{R}^2 \\ y \in \mathbb{R}^\times_{>0} \end{matrix} \right\}$ with
metric $\dfrac{dx_1^2 + dx_2^2 + dy^2}{y^2}$

is symm space of $SL_2(\mathbb{C})$ in Iwasawa co-ord

now
$$N = \left\{ \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} : z \in \mathbb{C} \right\}, \quad A = \left\{ \begin{pmatrix} \sqrt{a} & \\ & 1/\sqrt{a} \end{pmatrix} \right\}$$

(in general $O(n,1)$ not isogenous to another group)

$O(2,1) \sim SL_2(\mathbb{R})$, $O(5,1) \sim SL_2(\mathbb{H})$
$O(3,1) \sim SL_2(\mathbb{C})$
$O(4,1) \sim \dots$

in $O(5,1)$ $\quad N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{H} \right\}$ Hamilton's quat.

$A = \left\{ \begin{pmatrix} \sqrt{a} & \\ & 1/\sqrt{a} \end{pmatrix} \right\}$

$M = Z_K(A) = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \;\middle|\; \begin{matrix} a, b \in \mathbb{H} \\ \|a\| = \|b\| = 1 \end{matrix} \right\}$

$\left(\mathbb{H} = \{ x + iy + jz + kw \}, \; |\cdot| = x^2 + y^2 + z^2 + w^2 \right)$

here $M = SU(2) \times SU(2) \subset O(5) \subset SO(5,1)$
max'l cpt.

Again: $\mathfrak{a} \subset \mathfrak{p}$ max'l abelian, $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$

$\mathfrak{g} = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \oplus \mathfrak{a} \oplus \mathfrak{m}$

$\mathfrak{n} = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha \quad , \quad \mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$

Prop: A is a closed subgp, $\exp : \mathfrak{a} \to A$ is a diffeomorphism.

Pf: Know $\exp : \mathfrak{a} \to A$ is surjective since A is abelian. Also $K \times \mathfrak{p} \to G$ $\quad k \cdot \exp X$ is a diffeo, and $\mathfrak{a} \subset \mathfrak{p}$ is a closed subset.
So $\exp : \mathfrak{a} \to G$ maps $\mathfrak{a}$ to A bijectively & A is closed.

(Another proof: use action on $\mathfrak{g}$: $\bigcap_{\alpha \in \Sigma} \ker(\alpha) = \mathfrak{z}_{\mathfrak{g}} = \{0\}$

so $\|H\|_\infty \overset{\text{def}}{=} \max\{|\alpha(H)|\}_{\alpha \in \Sigma}$ is a norm on $\mathfrak{a}$

the e.v. of $\mathrm{Ad}_{\exp H}$ on $\mathfrak{g}$ are exactly $\{e^{\alpha(H)}\}_{\alpha \in \Sigma}$

so for any operator norm in $\mathrm{End}(\mathfrak{g})$,

$\|\mathrm{Ad}_{\exp H}\| \geq \exp(\|H\|_\infty)$

so if $H \to \infty$ in $\mathfrak{a}$, $\exp H \to \infty$ in A.

$\Rightarrow \ker \exp_{\mathfrak{a}}$ is trivial,

$\Rightarrow$ Image of a loc cpt set by proper map is closed.)

<u>Lemma:</u> If $X_\alpha \in \mathfrak{g}_\alpha$, $\neq 0$, then $H_\alpha = [X_\alpha, \theta(X_\alpha)] \neq 0$

<u>Pf:</u>

$H_\alpha \in \mathfrak{g}_{\alpha+(-\alpha)} = \mathfrak{g}_0$, $\theta(H_\alpha) = [\theta(X_\alpha), X_\alpha] = -H_\alpha$

so $H_\alpha \in (\mathfrak{a} \oplus \mathfrak{m}) \cap \mathfrak{p} = \mathfrak{a}$.

Pf: If $H \in \mathfrak{a}$ then,

$$B([X_\alpha, \theta(X_\alpha)], H) =$$

$$= B(\mathrm{ad}_{X_\alpha} \cdot \theta(X_\alpha), H) = B(\theta X_\alpha, [H, X_\alpha])$$

$$= \alpha(H)\, B(\theta X_\alpha, X_\alpha) = \alpha(H) \cdot B_\theta(X_\alpha, X_\alpha) \neq 0$$

choose H s.t. $\alpha(H) \neq 0$ neg. def.

$\blacksquare$

Prop: N is a closed subgp, $\exp: \mathfrak{n} \to N$ is a diffeo.

<u>Key idea:</u> A acts on N: $a \exp(X) a^{-1} = \exp(\mathrm{Ad}_a \cdot X)$

if $X \in \mathfrak{n}$, since roots are real on A, positive can find $H \in \mathfrak{a}$ s.t. $a = \exp H$ <u>~~conformly~~</u> expanding

so if something is true in a small nbd $\mathcal{U} \subset \mathfrak{n}$ then after applying $\mathrm{Ad}_{\exp tH} \in \mathrm{Aut}(\mathfrak{g})$, same thing is true in the large nbd $\exp(tH) \cdot \mathcal{U}$.

Pf: Let $\mathcal{U} \subset \mathfrak{n}$ be an open nbd of 0 s.t. $\exp: \mathcal{U} \to N$ is a diffeo onto the image

Fix $H \in$ positive Weyl chamber (s.t. $\alpha(H) > 0$ for $\alpha \in \Sigma^+$)

Suppose $\exp X = \exp Y$ for some $X, Y \in n$
Write $X = \sum_{\alpha > 0} X_\alpha$, $Y = \sum_{\alpha > 0} Y_\alpha$, $X_\alpha, Y_\alpha \in \mathfrak{g}_\alpha$.

Then $\mathrm{Ad}_{\exp(-tH)} \exp(X) = \exp(\mathrm{Ad}_{\exp(-tH)} X)$

$= \exp(\mathrm{Ad}_{\exp(-tH)} \sum_\alpha X_\alpha) = \exp(\sum_\alpha e^{-t\alpha(H)} X_\alpha)$

$\Rightarrow \exp(\sum_\alpha e^{-t\alpha(H)} X_\alpha) = \exp(\sum_\alpha e^{-t\alpha(H)} Y_\alpha)$

But for t large enough both sides are in U

thus $\sum_\alpha e^{-t\alpha(H)} X_\alpha = \sum_\alpha e^{-t\alpha(H)} Y_\alpha$

since $n = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ get $e^{-t\alpha(H)} X_\alpha = e^{-t\alpha(H)} Y_\alpha$
so $X_\alpha = Y_\alpha$, $X = Y$

For surjectivity, know: $\bigcup_{k=1}^{\infty} (\exp(U))^k$ is an open subgp
so all of N, so suppose
$g \in N$ has form $g = \exp(X_1) \cdots \exp(X_k)$
with $X_j \in U$.

Apply $\mathrm{Ad}_{\exp(-tH)}$ to get

$$\exp(-tH)\, g\, \exp(tH) = \prod_{j=1}^{n} \exp(\mathrm{Ad}_{\exp(-tH)} \cdot X_j)$$

$\exists$ small nbd $V_k \subset \mathcal{U}$ st $(\exp V_k)^k \subset \exp(\mathcal{U})$
(continuity of mult in N)

If t is large enough $\mathrm{Ad}_{\exp(-tH)} X_j \in V_k$

then $\exp(-tH)\, g\, \exp(tH) \in \exp(\mathcal{U})$

so $g \in \exp(\mathrm{Ad}_{\exp(tH)} \mathcal{U}) \subset \exp \mathfrak{n}$.

Get: $\exp_N : \mathfrak{n} \to N$ is a smooth bijection

Diffeo on $\mathcal{U}$, hence on $\bigcup_{t>0} \mathrm{Ad}_{\exp(tH)} \mathcal{U} = \mathfrak{n}$.

Let $\bar{N}$ = top. closure of N, closed ctd subgp of G.
Still normalized by A, so $\mathrm{lie}(\bar{N})$ decomposes into
irreps of A:

$$\mathrm{lie}(\bar{N}) = (\mathrm{lie}(\bar{N}) \cap \mathfrak{g}_0) \oplus \bigoplus_{\alpha} (\mathrm{lie}\, \bar{N} \cap \mathfrak{g}_\alpha)$$

if $g \in N$ then $\mathrm{ad}(g) = \mathrm{ad}(\exp(X))$, $X \in \mathfrak{n}$,
$= \exp(\mathrm{ad}\, X)$.

$\mathrm{ad}\, X$ is nilpotent (raises weights) so $\mathrm{ad}(g)$ is

unipotent (all e.v. $=1$). This is a closed subset
(matrices M: $(M-\mathrm{Id})^{\dim\mathfrak{g}}=0$)

Let $H \in \mathrm{Lie}(\bar{N}) \cap \mathfrak{g}_0$

$H \in \mathfrak{g}_0$ so $\mathrm{ad}\,H$ is semisimple, so $\exp(t\,\mathrm{ad}H)$ is semisimple + unipotent $\Rightarrow$ id.

$\Rightarrow \exp(tH) \in Z(G) \Rightarrow H \in Z_{\mathfrak{g}} = \{0\}$

Similarly let $X_{-\beta} \in \mathrm{Lie}(\bar{N}) \cap \mathfrak{g}_{-\beta}$, $\beta > 0$
then in basis $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$, $\mathrm{ad}\,X_{-\beta}$ is strictly lower-triangular, so

$\exp(t\,\mathrm{ad}\,X_{-\beta})$ is lower-triangular unipotent in this basis,
but $N \in$ upper-triangular unipotents

so $\exp(t\,\mathrm{ad}\,X_{-\beta}) \in$ diagonal unipotent $=\mathrm{Id}$
so $X_{-\beta}=0$.

$\Rightarrow \mathrm{Lie}(\bar{N}) \subset \mathrm{Lie}(N)$, so $\mathrm{Lie}(\bar{N}) = \mathrm{Lie}(N)$
both ctd $\Rightarrow \bar{N} = N$, N closed.

$\blacksquare$

Cor: The semidirect pdt $NA \subset G$ is closed, diffeomorphic to $\mathfrak{n} \oplus \mathfrak{a}$
(need to show: if $n_k \in N$ or $a_k \in A$ go to ∞, $n_k a_k \to \infty$)

In "usual" (explicit) groups, $G \subseteq GL_n(\mathbb{R})$

A = diagonal matrices of positive entries
N = upper triangular unipotents,

so obviously closed subgps

<u>Not general</u>: if G covering of $SL_2(\mathbb{R})$
$(\pi_1(SL_2(\mathbb{R})) = \pi_1(SO(2)) = \mathbb{Z})$

Any hom $f: G \to GL_n(\mathbb{R})$ is a rep'n of Lie $C = sl_2\mathbb{R}$, $\Rightarrow$ direct sum of irreps, all integrate to rep'ns of $SL_2(\mathbb{R})$,

so f factors through covering map
$$\begin{tikzcd} G \arrow[d] \arrow[dr, bend left, "f"] & \\ SL_2(\mathbb{R}) \arrow[r, dashed] & GL_n(\mathbb{R}) \end{tikzcd}$$

so G not isom to any subgp of $GL_n(\mathbb{R})$