Math 535, lecture 34, 3/4/2023

NO CLASS WED 5/4

Last time: Cartan involutions

$\mathfrak{g}$ real s.s. Lie alg then $\exists\, \theta \in \mathrm{Aut}(\mathfrak{g})$ s.t.
$B_\theta(X,Y) = B(X, \theta Y)$ is negative-definite

(proof: use $\mathfrak{g}_{\mathbb{C}}$)

Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ $\quad$ $+,-$ eigenspaces of θ

$\mathrm{ad}(\theta X) = -(\mathrm{ad}\, X)^*$ ← adjoint wrt B_θ

Cor: $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ is a Lie subalg closed under $*$

Thm: G ctd s.s. θ as above
(1) $\exists\, \Theta \in \mathrm{Aut}(G)$ with $d\Theta = \theta$.
(2) $G^\Theta = K$ is the subgp with Lie alg. $\mathfrak{k}$.
 K is closed, contains $Z = Z(G)$, K/Z is cpt.
(3) $K \times \exp \mathfrak{p} \to G$ is a diffeo. ("polar decomposition")

Example: $G = GL_n(\mathbb{R})$, $\Theta(g) = {}^t g^{-1}$, $K = O(n)$

$\mathfrak{k}$ = anti-symmetric matrices

$\mathfrak{p}$ = symmetric matrices

$\exp \mathfrak{p}$ = pos. def. symm matrices

identify $\mathfrak{g}$, $\operatorname{ad}\mathfrak{g}$ b

Pf: Start with $\bar{G} = \operatorname{Ad}(G) \subset GL(\mathfrak{g})$.

Equip $\mathfrak{g}$ with inner pdt B_θ. If $g \in \bar{G}$, $X, Y \in \mathfrak{g}$

Then $[gXg^{-1}, gYg^{-1}] = gXYg^{-1} - gYXg^{-1}$
$= g[X,Y]g^{-1}$

take * set

$$[g^{*-1}Y^*g^*, g^{*-1}X^*g^*] = g^{*-1}[Y^*, X^*]g^*$$

$\Rightarrow (g^*)^{-1} \in \operatorname{Aut}(\mathfrak{g})$ but $g \in \bar{G} = \operatorname{Aut}(\mathfrak{g})^\circ$
so $g^* \in \bar{G}$.

Set $\bar{\Theta}(g) = (g^*)^{-1}$. (restriction of Cartan involution of $GL(\mathfrak{g})$ to $\bar{G}$)

Clearly $\bar{\Theta} \in \operatorname{Aut}(\bar{G})$, $\bar{\Theta}^2 = \mathrm{Id}$, $d\bar{\Theta} = \theta$.

$\bar{K} = \bar{G}^\Theta = \bar{G} \cap GL(\mathfrak{g})^\Theta = \bar{G} \cap O(B_\theta)$ is cpt with Lie alg $\mathfrak{g} \cap$ fixed pts of t^{-1} $= \mathfrak{g}^{\theta} = \mathfrak{k}$.

The map $\bar{K} \times \mathfrak{p} \to \bar{G}$
$(k, H) \mapsto k\exp(H)$
is smooth.

Suppose $g = k\exp(H)$. Then $g^*g = \exp(H)^* k^* k \exp(H)$

$k^* = k^{-1}$, $H^* = H$ ($\theta(H) = -H$ def of $\mathfrak{p}$)

so $g^*g = \exp(2H)$, $H = \frac{1}{2}\log(g^*g)$

map $g \mapsto H$ is smooth, then $k = g\exp(-H)$.

(shows that map is injective, will be diffo if inverse is defined everywhere)

<u>Lemma</u>: Since g^*g is symm pos def, it is of form $\exp(2H)$, $H \in \mathfrak{g}$.

So define $k = g\exp(-H)$ then $k^*k = \underbrace{\exp(-H)\,\underbrace{g^*g}_{\exp(2H)}\exp(-H)}_{= id}$

so $k \in \bar{K}$.

On G, $Z_{\mathfrak{g}} = \{0\}$ ($\mathfrak{g}$ is s.s.) so $\dim Z = 0$, Z is closed so discrete, then $\mathrm{Ad}: G \to \bar{G}$ is the covering map $G \to G/Z$.

Set K = inverse image of $\bar{K}$. Contains Z, is closed, $K/Z = \bar{K}$ is cpt.

$q : G/K \to \bar{G}/\bar{K}$ the quotient map, cts.

surjective since Ad is, injective $Z \subset K$

to show inverse is cts suppose $\mathrm{Ad}(g_n)\bar{K} \to \mathrm{Ad}(g)\bar{K}$ want $g_n K \to gK$ in G/K.

Since $\bar{K}$ is cpt may assume $\mathrm{Ad}(g_n) \to \mathrm{Ad}(g)$.
Then $\mathrm{Ad}(g^{-1} g_n) \to 1$ in $\bar{G}$. Since Ad is a cover, have $1 \in \bar{\mathcal{U}} \subset \bar{G}$ s.t. $\mathrm{Ad}^{-1}(\bar{\mathcal{U}}) = Z \times \mathcal{U}$ for nbd $1 \in \mathcal{U} \subset G$ s.t. $\mathrm{Ad}|_{\mathcal{U}} : \mathcal{U} \to \bar{\mathcal{U}}$ is a homeo. Write

$$g_n = u_n z_n \quad u_n \in \mathcal{U},\ z_n \in Z$$
$$g = u z \quad u \in \mathcal{U} \quad z_n \in Z$$

then $g^{-1} g_n = (u^{-1} u_n) \cdot (z^{-1} z_n)$ and $u^{-1} u_n \to 1$
so $u_n \to u$. $\Rightarrow$ $u_n ZK \Rightarrow uZK \Leftrightarrow g_n K \to gK$.

For $g \in G$ have $Ad(g) = Ad(k) \cdot \exp(\operatorname{ad} H)$ for some $k \in K$, $H \in \mathfrak{p}$. $\Rightarrow g = kz \exp H$ for $z \in Z$ and $kz \in K$. So $G = K \cdot \exp(\mathfrak{p})$

Uniqueness of H follows from uniqueness in $\bar{G}$. Then $k = g\exp(-H)$ is unique too.
Still local diffeo so global diffeo

$\Rightarrow G \cong K \times \mathfrak{p}$, K is a deformation retract of G.
$\Rightarrow K$ ctd, $\pi_1(G) = \pi_1(K)$

Eg. $\pi_1(SL_2(\mathbb{R})) = \pi_1(SO(2)) \cong \mathbb{Z}$

Let $\tilde{G}$ be the universal covering group of G. Everything applies here, get $\tilde{K}$ covering K, $\bar{K}$. $\Theta \in \mathrm{Hom}_{Lie}(\mathfrak{g}, \mathfrak{g})$ extends to $\tilde{\Theta}: \tilde{G} \to \tilde{G}$, covering $\bar{\Theta}$.

Since θ is trivial on k, $\tilde{\Theta}$ is trivial on $\tilde{K}$, so on $Z \subset \tilde{K}$. Now $\tilde{\Theta}$ descends to $G = \tilde{G}/_{\text{central subgp}}$ with fixed pts K.

QED