Math 535, lecture 33, 31/3/2023

Last time: $\mathfrak{g}$ s.s. Lie alg. $\iff B(X,Y) = \mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$ is non-degenerate

$\Rightarrow \mathrm{ad}_X \neq 0$ for all $X \in \mathfrak{g}$

G s.s. if Lie $\mathfrak{g}$ is

Saw: $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathfrak{g}$. $\Rightarrow \mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(G)$

Today: Real forms and Cartan involutions

Let $\mathfrak{g}$ be a real s.s. Lie alg. Then $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ is a complex s.s. Lie alg $\Rightarrow$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}};\mathfrak{h})} \mathfrak{g}_\alpha$$

$\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ Cartan subalgebra, $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$

Fact: Can choose generators $X_\alpha \in \mathfrak{g}_\alpha$ s.t.

(1) $H_\alpha = [X_\alpha, X_{-\alpha}] \in \mathfrak{h}$ is the coroot.

(2) If $\alpha+\beta$ is a root, $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ with $N_{\alpha,\beta} \in \mathbb{R}$, $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$.

Cor: Set $\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \forall \alpha: \alpha(H) \in \mathbb{R}\} = \mathrm{Span}_{\mathbb{R}}\{H_\alpha\}_{\alpha \in \Phi}$

Then $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha} \mathbb{R} X_\alpha$ is a real subalgebra of $\mathfrak{g}$,

the split real form (form: $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0 = \mathfrak{g}_{\mathbb{C}}$)

Cor: Let $\mathfrak{u} = i\mathfrak{h}_0 \oplus \bigoplus_{\alpha > 0} \mathbb{R}(X_\alpha - X_{-\alpha})$

$$\oplus \bigoplus_{\alpha > 0} i\mathbb{R}(X_\alpha + X_{-\alpha})$$

Then $\mathfrak{u}$ is a real form with negative-definite Killing form, that is a compact real form.

Lemma: Let $\tau: \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be complex conjugation wrt $\mathfrak{u}$. (if $X, Y \in \mathfrak{u}$ then $\tau(X + iY) = X - iY$)

Then $\tau([X,Y]) = [\tau(X), \tau(Y)]$ for all $X, Y \in \mathfrak{g}_{\mathbb{C}}$.

Furthermore let $\tilde{B}$ be the Killing form of $\mathfrak{g}_{\mathbb{C}}$ thought of as a real Lie algebra. Then

$(X,Y) \mapsto \tilde{B}(X, \tau(Y))$ is negative definite.

Pf: (direct calculation from $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u} \oplus_{\mathbb{R}} i\mathfrak{u}$ as real Lie algebras, $B_{\mathfrak{u}}$ is negative definite.

<u>Def:</u> Let $\mathfrak{g}$ be a real Lie algebra. An involution $\Theta \in \mathrm{Aut}(\mathfrak{g})$ is a Cartan involution if $B_{\mathfrak{g}}(X, \Theta Y)$ is negative definite

<u>Example:</u> $\mathfrak{g} = \mathfrak{gl}_n \mathbb{R}$, $\Theta(X) = -X^t$. Then

$$[X,Y]^t = (XY)^t - (YX)^t = [Y^t, X^t] = -[(-X^t), (-Y^t)]$$

$$B_\Theta(X,Y) = B(X, \Theta Y) = -\sum_{i,j=1}^{n} X_{ij} Y_{ij}.$$

<u>Lemma:</u> $g \in \mathrm{Aut}(\mathfrak{g})$ be symmetric, pos. def. wrt B_Θ. Then $g = \exp(\mathrm{ad}\, X)$ for some $X \in \mathfrak{g}$. (and $g \in \mathrm{Aut}(\mathfrak{g})^\circ$.)

<u>Pf:</u> By spectral theorem $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ where

$$g\restriction_{\mathfrak{g}_\lambda} = \lambda. \quad \text{Then } [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda\mu}.$$

So if $t > 0$, $X \in \mathfrak{g}_\lambda$, $Y \in \mathfrak{g}_\mu$, then

$$[g^t X, g^t Y] = \lambda^t \mu^t [X,Y] = g^t[X,Y]$$

so $g^t \in \mathrm{Aut}(\mathfrak{g})$, so $t \mapsto g^t$ is a 1-param subgp in $\mathrm{Aut}(\mathfrak{g})$, so of form $\exp(t \cdot \mathrm{ad}\, X)$.

<u>Theorem:</u> Every real semisimple Lie algebra has a Cartan involution

<u>Idea:</u> $\mathfrak{g}$ is real ss. write $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}^{\mathbb{R}}$ then $\mathfrak{g}$ = fixed points of involution $\sigma \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}^{\mathbb{R}})$ which is complex conjugation wrt $\mathfrak{g}$.

$\mathfrak{g}_{\mathbb{C}}$ viewed as real Lie alg

Show if $\mathfrak{g}_{\mathbb{C}}^{\mathbb{R}}$ has Cartan involution, $\mathfrak{g}$ = fixed points of involution, then $\mathfrak{g}$ has Cartan involution too.

<u>Cor:</u> Equip $\mathfrak{g}$ with inner pdt $B_\theta(X,Y) = -B(X,\theta Y)$

then $(\mathrm{ad}\, X)^* = -\mathrm{ad}(\theta X) \Rightarrow \underbrace{\mathrm{ad}\,\mathfrak{g}}_{\cong \mathfrak{g}} \subset GL(\mathfrak{g})$ is closed under transpose.

<u>Pf:</u> write $\mathfrak{k}, \mathfrak{p}$ for the $+1$, -1 eigenspaces of Θ.

($\mathfrak{g} = \mathfrak{gl}_n \mathbb{R}$, $\Theta(X) = -X^*$,

$\mathfrak{k}$ = antisymmetric matrices = Lie $(O(n))$
$\mathfrak{p}$ = symmetric matrices)

so $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$

<u>Observe:</u> $\mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{g} \oplus i\mathfrak{g}$ is a compact real form.

For example, $\mathfrak{k} \oplus i\mathfrak{p}$ = anti-Hermitian matrices
= Lie $(U(n))$

Cartan decomposition

On Lie algebra level have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
What happens for G?

For $G = GL_n(\mathbb{R})$, let $\Theta(g) := {}^t g^{-1} \in \mathrm{Aut}(GL_n(\mathbb{R}))$

then $d\Theta = \theta$, fixed pts of $\Theta = O(n)$ = max'l cpt.

<u>Thm:</u> G ctd s.s. Lie gp, $\theta \in \mathrm{Aut}(\mathfrak{g})$ a Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then:

(1) There exists an involution $\Theta \in \mathrm{Aut}(G)$ s.t. $d\Theta = \theta$.

(2) $G^{\Theta} = K$ is the Lie subgp with Lie alg. $\mathfrak{k}$.
K is closed, contains $Z = Z(G)$, K/Z is cpt.

(3) Polar decomposition $G = K \exp \mathfrak{p}$ is a diffeo.

(4) (When Z is finite, K is a max'l cpt subgp)

Pf: Start with $\bar{G} = \mathrm{Ad}(G)$ (same Lie algebra!)
Equipping $\mathfrak{g}$ with inner prdt B_θ, if $g \in \mathrm{Aut}(\mathfrak{g})$ then for $X, Y \in \mathfrak{g}$,

$$[gXg^{-1}, gYg^{-1}] = g[X, Y]g^{-1}$$

(identity of linear maps in $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$)
Apply θ, i.e. take transpose

$$\Rightarrow ({}^t g^{-1}) \in \mathrm{Aut}(\mathfrak{g}), \text{ so } \mathrm{Aut}(\mathfrak{g}) \text{ is closed}$$

under $\bar{\Theta}(g) = {}^t g^{-1}$, $d\bar{\Theta} = \theta$.

Since $\bar{G} = \mathrm{Aut}(\mathfrak{g})^{\circ}$, get Cartan involution of $\bar{G}$.