Math 535, Lecture 25 13/3/2023

Last time: $\mathfrak{g} = \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}_2\mathbb{C} = \mathrm{span}\{e, f, h\}$

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Then every irred n-dim rep'n $(\ell = \frac{n-1}{2} \in \frac{1}{2}\mathbb{Z})$ has the form $\mathrm{span}\{v_m\}_{m=-\ell}^{\ell}$ with

$$\begin{cases} \pi(h)\underline{v}_m = 2m v_m & \\ \pi(f)v_m = \underline{v}_{m-1} & (\pi(f)\underline{v}_{-\ell} = \underline{0}) \\ \pi(e)\underline{v}_m = (\ell-m)(\ell+1+m)v_{m+1} & (\pi(e)\underline{v}_\ell = \underline{0}) \end{cases}$$

(in general restrict $(\pi, V) \in \mathrm{Rep}(G;\mathbb{C})$ to T to get $V = \bigoplus_{\mu \in \Lambda^*} V_\mu$, where $\mathfrak{g}$ acts by $\pi(\mathfrak{g}_\alpha) \cdot V_\mu \subset V_{\mu+\alpha}$)

Call $\pi(e), \pi(f)$ ladder operators.
Think of rep'n as generated by the highest weight vector v_ℓ by the lowering operator $\pi(f)$

Cor: Exists at most one rep'n of each dim (up to isom)

Thm: There is such a rep'n - check commutation relations

e.g. $[\pi(h), \pi(e)] = \pi(2e)$

Cor: ($SL_2(\mathbb{C})$, $SU(2)$ are simply connected) These representations of the Lie algebra integrate to representations of the group. In particular $s_\alpha \in W(SU(2); T)$ acts

2nd Pf of existence: Let $\mathbb{C}[x,y]$ be the ring of polynomials, on which $SL_2(\mathbb{C})$ acts by change of variables:

$$(g \cdot P)\begin{pmatrix} x \\ y \end{pmatrix} = P\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Let $V_n = \{ P \in \mathbb{C}[x,y] \mid P \text{ homog. of deg } n-1 \}$

$$= \mathrm{Span}_{\mathbb{C}} \{ x^i y^j \mid i+j = n-1 \}$$

clearly $SL_2(\mathbb{C})$-inv't.

$g = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} = \exp(ih)$ then

$$g \cdot (x^k y^\ell) = (e^{-i\theta} x)^k (e^{i\theta} y)^\ell = e^{(\ell - k) i\theta} x^k y^\ell$$

so weights of V_n range from $2\frac{n-1}{2}$ to $-2\frac{n-1}{2}$

$\Rightarrow$ irred rep'n with hiegst weight $\ell = \frac{n-1}{2}$.

(alternative: subrep'n is spanned by a subset of the monomials by torus action, use unipotents $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$ to see each monomial generates all others)

Also: When n is odd, $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ acts trivially on V_n

When n is even, $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ acts by -1.

$\Rightarrow$ $\mathrm{Irr}(SO(3))$ are $\{V_n\}_{n \text{ odd}}$

Prop: Every f.d. rep'n of $\mathfrak{sl}_2\mathbb{C}$ is completely reducible

Pf 1: ("Weyl unitary trick") let $\pi : \mathfrak{sl}_2\mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$ be a rep'n. Restricting to $\mathfrak{su}(2)$ this integrates to a rep'n of $SU(2)$ since $SU(2)$ is SC

$\Rightarrow V$ = direct sum of irreps of $SU(2)$

Each irred subspace is $SU(2)$-inv't
$\Rightarrow su(2)$ inv't $\Rightarrow su(2)_{\mathbb{C}} = sl_2\mathbb{C}$ inv't.

(same argument works for $sl_2\mathbb{R}$, $SL_2(\mathbb{R})$) ∎

<u>Pf 2:</u> (Lie algebra method) Let $\mu \in \mathbb{C}$ be an eigenvalue of $\pi(h)$ on $V \Leftrightarrow$ weight of V.
Apply $\pi(e)$ k times find $\underline{v}_0$ with ev. $\lambda = \mu + 2k$, killed by $\pi(e)$. As before set $v_j = \pi(f)^j v_0$ have

$$\pi(h) v_j = (\lambda - 2j) v_j$$
$$\pi(e)\pi(f) v_j = (j+1)(\lambda - j) v_j$$

If $\lambda \neq j$ can keep lowering but V f.d. $\Rightarrow$ process stops

$$\Rightarrow \lambda \in \mathbb{Z}_{\geq 0}, \; \mu \in \mathbb{Z}$$

Now let λ be the highest weight in V, let $\{v_{\lambda, i}\}_{i=1}^{\dim V_\lambda}$ be a basis

Define $\underline{v}_{\lambda,i,j} = \pi(f)^j \underline{v}_{\lambda,i}$ $\quad (0 \leq j \leq 2\lambda)$

Then each $\mathrm{Span}\{\underline{v}_{\lambda,i,j}\}_{j=0}^{2\lambda}$ is an irrep., linearly indep (applying $\pi(e)^j$ will push minimal dependence into V_λ).

Since weight spaces are indep any dependence will have form

$$\sum_i a_i \underline{v}_{\lambda,i,j} = \underline{0} \qquad \text{(some fixed } j\text{)}$$

apply $\pi(e)^j$: $\pi(e)^j \underline{v}_{\lambda,i,j} = \alpha \cdot \underline{v}_{\lambda,i}$
where α depends on λ, j is nonzero
so set

$$\alpha \sum_i a_i \underline{v}_{\lambda,i} = \underline{0} \quad \text{so all } a_i = 0.$$

let U_λ = sum of these irreps

If $U_\lambda \neq V$ let λ' be the highest weight in V/U_λ, necessarily $\lambda' < \lambda$.

let $W_{\lambda'} = \{\underline{v} \in V_{\lambda'} \mid \pi(e)\underline{v} = \underline{0}\}$

claim: $W_{\lambda'} \oplus (\mathcal{U}_\lambda)_{\lambda'} = V_{\lambda'}$

Pf: If $\underline{v} \in (\mathcal{U}_\lambda)_{\lambda'}$, then $\pi(e)\underline{v} \in (\mathcal{U}_\lambda)_{\lambda'+2}$ is nonzero (raise vector of non-max'l weight)

so $(\mathcal{U}_\lambda)_{\lambda'} \cap W_{\lambda'} = \underline{0}$.

Consider $\tilde{\underline{v}} = \dfrac{\pi(f)\pi(e)\underline{v}}{j(\lambda-j+1)} \in (V_\lambda)_{\lambda'}$

Then $\pi(e)\tilde{\underline{v}} = \pi(e)\underline{v}$ (check in $\mathcal{U}_{\lambda'}$)

so $\tilde{\underline{v}} - \underline{v} \in W_{\lambda'}$.

As before pick basis of $W_{\lambda'}$, get direct sum of irreps with those highest weight vectors call this $\mathcal{U}_{\lambda'}$, also indep of $\mathcal{U}_\lambda$

If $V \neq \mathcal{U}_\lambda \oplus \mathcal{U}_{\lambda'}$ let λ'' be the highest remaining weight; check

$W_{\lambda''} = \{\underline{v} \in V_{\lambda'} \mid \pi(e)\underline{v} = 0\}$ how

$$V_{\lambda''} = W_{\lambda''} \oplus (U_{\lambda})_{\lambda''} \oplus (U_{\lambda'})_{\lambda''}$$

continue by induction.

<u>Cor</u>: Every f.d. rep'n of $\mathfrak{sl}_2\mathbb{C}$ is a sum of weight spaces $\Leftrightarrow$ $\pi(\mathfrak{h})$ is diagonalizable

<u>Remark</u>: Arguments would work for a general algebra/grp of semisimple rank 1

<u>Next lecture</u>: $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ = "universal enveloping algebra":
associative algebra generated by $\mathfrak{g}_{\mathbb{C}}$ subject to $XY - YX = [X,Y]$ for $X, Y \in \mathfrak{g}_{\mathbb{C}}$

Aside: G loc. cpt; (π, V) "reasonable" rep
$f \in C_c(G)$

$$\pi(f)\cdot \underline{v} = \int_G f(g)\,\pi(g)\,\underline{v}\, dg$$