Math 535, lecture 22 6/3/2023

Last time: Weyl chambers

Combinatorial structure on $\mathfrak{t} = \mathrm{Lie}\, T$:
for $f : \Phi \to \{+,-,0\}$ set

$$C_f = \{ H \in \mathfrak{t} \mid \forall \alpha : \operatorname{sgn}(\alpha(H)) = f(\alpha) \}$$

clearly convex cones (closed under $tx + sy$, $t, s > 0$)

$\mathfrak{t} = \coprod_f C_f$, of which the open ones are given by fcns $f : \Phi \to \{+,-\}$, jointly the complement of $\bigcup_\alpha \mathcal{U}_\alpha$. ($\mathcal{U}_\alpha = \ker \alpha$)

Say C_g is facet of C_f if $C_g \subset \overline{C_f}$.
(C_f is open in $\overline{C_f}$, complement lower-dim)

Maximal facets of chambers are called walls, are the $\mathcal{U}_\alpha$.

$\Delta = \{ \alpha \mid \mathcal{U}_\alpha$ is wall of fund chamber $C \}$

Def: $W' = \langle \{ s_\alpha \mid \alpha \in \Delta \} \rangle \subset \langle \{ s_\alpha \mid \alpha \in \Phi \} \rangle \subset W$

Saw: W' acts transitively on chambers
$\mathrm{Stab}_W(C) = \{1\}$

Cor: $W' = W$ in bijection with chambers

$$\mathrm{Stab}_W(H) = \langle \{ s_\alpha \mid \alpha(H) = 0 \} \rangle$$

Today: Geometry of root systems
$\Longleftrightarrow$ structural info on $\mathfrak{g}$. $\Rightarrow G$.

Observation: Since s_α fixes $\mathfrak{u}_\alpha$, $s_\alpha - \mathrm{id}_{\mathfrak{t}}$ vanishes on $\mathfrak{u}_\alpha$ $\Rightarrow$ has rk 1, factors through α

$\Rightarrow (s_\alpha - \mathrm{id})(x) = -\alpha(x) \cdot \check{\alpha}$ for some $\check{\alpha} \in \mathfrak{t}$.

$\Rightarrow \quad s_\alpha(x) = x - \alpha(x)\check{\alpha}$

On $\mathfrak{t}^*$, $\quad s_\alpha(\nu) = \nu - \nu(\check{\alpha})\alpha$

since $\quad \langle s_\alpha^*(\nu), x \rangle = \langle \nu, s_\alpha \cdot x \rangle$
$= \langle \nu, x - \alpha(x)\check{\alpha} \rangle = \langle \nu, x \rangle - \nu(\check{\alpha}) \langle \alpha, x \rangle$

Since $s_\alpha(\alpha) = -\alpha$ have $\alpha(\check{\alpha}) = 2$.

<u>Defn</u>: Call $\check{\alpha}$ the coroot assoc. to α.
$\check{\Phi} \subset \mathfrak{t}$ for the set of coroots

<u>Warning</u>: Even if $\alpha+\beta$ is a root, need not have $(\alpha+\beta)^\vee = \check{\alpha} + \check{\beta}$. Root systems $\Phi, \check{\Phi}$ need not be isomorphic.

<u>Ex</u>: $(\mathfrak{t}/_3, \check{\Phi})$ is a root system ("dual root system")

<u>Lemma</u>: Coroots are integral: $\check{\alpha} \in \Lambda = \mathrm{Ker}(\exp_T)$

<u>Pf</u>: Consider $h = \exp(\frac{1}{2}\check{\alpha})$. Then

$$\chi_\alpha(h) = \exp(2\pi i \alpha(\tfrac{1}{2}\check{\alpha})) = \exp(2\pi i) = 1$$

$\Rightarrow h$ is Ad_{s_α}-stable,

$s_\alpha(\check{\alpha}) = -\check{\alpha}$ so $\mathrm{Ad}(s_\alpha) \cdot h = \exp(-\frac{1}{2}\check{\alpha}) = h^{-1}$.

so $h^2 = \exp(\check{\alpha}) = 1$.

Cor: Since $\Phi \subset \Lambda^*$, $\beta(\alpha^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$

Notation: $n_{\alpha\beta} = \beta(\alpha^\vee)$

Called Cartan numbers of $\mathfrak{g}$.

$(n_{\alpha\beta})_{\alpha,\beta \in \Delta}$ called the Cartan matrix.

(Look up "Kac-Moody Lie algebras" for more)

Note: $s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha = \beta - n_{\alpha\beta} \cdot \alpha$

Def: The coroot lattice is the subgroup $\Gamma \subset \Lambda$ generated by the coroots

Fact: $\Lambda / \Gamma \cong \pi_1(G)$.

Cor: $\tilde{G}$ is cpt iff $\pi_1(G)$ finite iff $\mathrm{rk}\, \Gamma = \mathrm{rk}\, \Lambda$ iff $\Phi^\vee$ span $\mathfrak{t}$, iff $\mathfrak{z} = 0$ iff $Z(G)$ is finite.

In each case say G is semisimple.

Fact: G is semisimple iff it's the almost direct pdt of nonabelian almost simple groups

(E.g. $SU(2) \times SU(2) / \{(I, I), (-I, -I)\}$)

Equipping $\mathfrak{t}$ with inv't inner pdt; same is true for $\mathfrak{t}^*$. Then

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$$

so under identification $\mathfrak{t} \leftrightarrow \mathfrak{t}^*$, $\alpha^\vee \leftrightarrow \alpha^{\vee *} = \frac{2\alpha}{\langle \alpha, \alpha\rangle}$

Now $n_{\alpha\beta} = \beta(\alpha^\vee) = \langle \beta, \alpha^{\vee *}\rangle = 2\frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}$

$$\Rightarrow n_{\alpha\beta} n_{\beta\alpha} = 4\frac{\langle \alpha, \beta\rangle^2}{\langle \alpha, \alpha\rangle\langle \beta, \beta\rangle} \underset{C-S}{\leq} 4$$

(equality iff α, β are proportional)

$n_{\alpha\beta}, n_{\beta\alpha} \in \mathbb{Z}$, both zero if $\alpha \perp \beta$, nonzero otherwise, their pdt is positive, in $\{1, 2, 3\}$ $(\alpha \neq \pm\beta)$, if $\alpha \neq \pm\beta$, pair $(n_{\alpha\beta}, n_{\beta\alpha})$ is one of

$$(0,0), \quad \pm(1,1), \quad \pm(1,2), \quad \pm(1,3)$$

Thus pair $n_{\alpha\beta}, n_{\beta\alpha}$ determines angle between α, β

if $\alpha \neq \beta$ also ratio of lengths

<u>Cor:</u> let $\alpha \neq \pm\beta$, $n_{\alpha\beta} > 0$ (equiv $\langle \alpha, \beta \rangle > 0$)
Then $\alpha - \beta \in \Phi$

<u>Pf:</u> Wlog $n_{\beta\alpha} = 1$ (else $n_{\alpha\beta} = 1$) then

$$s_\beta(\alpha) = \alpha - n_{\beta\alpha} \cdot \beta = \alpha - \beta.$$

Fixing fund chamber C, call $\alpha \in \Phi$ <u>positive</u> if $\alpha|_C$ is positive, <u>negative</u> otherwise (write $\Phi = \Phi^+ \sqcup \Phi^-$)

<u>Def:</u> Call $\alpha \in \Phi^+$ simple if it's not a positive sum of positive roots, $\Delta = \{\text{simple roots}\}$

(both the notion of positivity & the simple system Δ depend on choice of C)

<u>Lemma:</u> Every positive root is a sum of simple roots
<u>Pf:</u> Fix $H \in C$, let α be a counter example, $\alpha(H)$ minimal

α not a simple root, so $\alpha = \sum_i \beta_i$, β_i positive roots. Then $\alpha(H) = \sum_i \beta_i(H)$ so $\beta_i(H) < \alpha(H)$. Then each $\beta_i(H)$ is a sum of positive roots, so α has same form.

Prop: $\Delta \subset t^*$ is linearly indep

Pf: let $\alpha, \beta \in \Delta$ be distinct. If $\langle \alpha, \beta \rangle > 0$ then either $\alpha - \beta$ or $\beta - \alpha$ would be a positive root; so either $\alpha = (\alpha - \beta) + \beta$ or $\beta = (\beta - \alpha) + \alpha$ contradict both being simple. Thus $\langle \alpha, \beta \rangle \leq 0$

Also $\Delta \subset \{ v \mid v(H) > 0 \}$

Suppose Δ is dependent. Can always write the dependence in form

$$\sum_{\alpha \in A} a_\alpha \cdot \alpha = \sum_{\beta \in B} b_\beta \cdot \beta$$

where $A, B \subset \Delta$ disjoint, $a_\alpha, b_\beta > 0$

Call this vector v.

Then

$$0 \leq \langle v, v \rangle = \sum_{\substack{\alpha \in A \\ \beta \in B}} a_\alpha b_\beta \langle \alpha, \beta \rangle \leq 0$$

so $v = 0$,

$\sum_\alpha a_\alpha \alpha = \sum_\beta b_\beta \beta = 0$. Evaluate at H:

$0 = \sum_\alpha a_\alpha \cdot \alpha(H)$, $\sum_\alpha b_\beta \beta(H) > 0$ all $a_\alpha, b_\beta > 0$ all $\alpha(H), \beta(H) > 0$

<u>Lemma:</u> Δ span $(\mathfrak{t}/\mathfrak{z})^*$
<u>Pf:</u> all roots vanish on $\mathfrak{z}$, lie in $(\mathfrak{t}/\mathfrak{z})^*$.
Conversely, span contains Φ, so the common kernel of the span acts trivially on $\bigoplus_\alpha \mathfrak{g}_\alpha \oplus \mathfrak{g}_0 = \mathfrak{g}$, so is $\mathfrak{z}$, span is $(\mathfrak{t}/\mathfrak{z})^*$.

<u>Cor:</u> $\#\Delta = \dim_{\mathbb{R}} (\mathfrak{t}/\mathfrak{z})^* = \dim_{\mathbb{R}} \mathfrak{t}/\mathfrak{z}$ = semisimple rank

<u>Lemma:</u> $\{U_\alpha\}_{\alpha\in\Delta}$ are walls of C.

<u>Pf:</u> $\{H \mid \forall \alpha\in\Delta : \alpha(H) > 0\} \underset{\Uparrow}{=} \{H \mid \forall \beta\in\Phi^+ : \beta(H) > 0\} = C$

$\beta\in\Phi^+ \Rightarrow \beta$ positive sum from Δ

since Δ indep they are the walls

<u>Def:</u> A system of simple roots is a subset $\Delta\subset\Phi$ s.t. each $\beta\in\Phi$ is a sum elements of Δ or the negative of such a sum

Cor: Each system of simple roots is the set of walls of a Weyl chamber, get bijection

$$\{\text{simple systems}\} \leftrightarrow \{\text{notions of positivity}\} \leftrightarrow \{\text{Weyl chambers}\}$$

Weyl group acts simply transitively on all.

(**Cor:** Every root belongs to a simple system)

Example: $G = U(N)$, $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_n \mathbb{C}$

roots $\alpha_{ij} = e_i - e_j$ $\qquad e_i(H) = H_i$

$H = \mathrm{diag}(H_1, \ldots, H_n)$

Take $H = (H_1 > H_2 > \ldots > H_n)$

Then $\alpha_{ij}(H) > 0$ if $i<j$, eg. $\alpha_{25} > 0$

$\alpha_{73} < 0$

take $\Delta = \{\alpha_{i,i+1}\}_{i=1}^{n-1}$.

Then if $i<j$ $\alpha_{ij} = e_j - e_i = (e_j - e_{j-1}) + (e_{j-1} - e_{j-2}) + \ldots (e_{i+1} - e_i)$

$= \sum_{k=i}^{j-1} \alpha_{k,k+1}$

Weyl chamber: $C = \{H \mid H_1 > H_2 > \dots > H_n\}$

intersect $\mathcal{U}_{13} = \{H_1 = H_3\}$ with $\bar{C}$

set:

$$\{H_1 = H_2 = H_3 \geq H_4 \geq \dots\}$$

$$\cap$$

$$\{H_1 = H_2 \geq H_3 \geq \dots\} \; \bigcap$$

$$\{H_1 \geq H_2 = H_3 \geq H_4 \geq \dots\}$$

joint kernel of roots is $\bigcap_{\alpha\in\Phi} \ker\alpha = \bigcap_{\alpha\in\Delta}\ker\alpha =$

$$= \{\operatorname{diag}(H, H, \dots, H)\} = \mathfrak{z}$$

Ex: V f.d. vsp, $\Phi \subset V^*$.

$$\operatorname{Span}\Phi = \{\alpha \mid \ker\alpha \supset \bigcap_{\varphi\in\Phi}\ker\varphi\}$$