Math 535, Lecture 20   1/3/2023

Last time: Classified rk 1 groups,

Idea: $\mathfrak{t} \subset \mathfrak{g}$ torus, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$

if β smallest positive root looked at

$$V = \mathbb{C} X_{-\beta} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$$

which was $X_\beta, X_{-\beta}$ - inv't, $\Rightarrow H_\beta$ - inv't

Then $\operatorname{Tr} H_\beta = 0$ forced $\dim X_\beta = 1$, $\mathfrak{g}_\alpha = 0$ if $\alpha > \beta$

For higher-rank groups, if α root set

$$\mathfrak{u}_\alpha = \ker \alpha \subset \mathfrak{t} \; ; \quad G_\alpha = Z_G(\mathfrak{u}_\alpha)$$

$$\overline{G}_\alpha = G_\alpha / \ker(\chi_\alpha) \quad ; \quad \chi_\alpha(\exp t H) = e(\alpha(tH))$$

image of T in $\overline{G}_\alpha$ is the 1d torus with Lie algebra $\mathfrak{t}/\mathfrak{u}_\alpha$. $\Rightarrow$ $\overline{G}_\alpha$ is rk 1, not commutative, so $\overline{G}_\alpha \cong SU(2)$ or $SO(3)$

<u>Prop:</u> G_α is ctd, s.s. rk 1

(1) $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha} = 1$

(2) $W(G_\alpha : T) \cong C_2$

(3) Let $s_\alpha \in W(G_\alpha : T) \subset W(G; T)$ be the nontrivial element. Then $Ad(s_\alpha)|_{\mathfrak{t}} \in GL(\mathfrak{t})$ is a reflection in the hyperplane $\mathcal{U}_\alpha$.

<u>Pf:</u> G_α = centralizer of torus $Ker(\chi_\alpha)$ so ctd

$T \subset G_\alpha$ still max'l torus, $\overline{G}_\alpha$ has rk 1 so G_α has s.s. rk 1

(1) Let β be a root proportional to α.

Then if $H \in \mathcal{U}_\alpha$, $\pm\beta(H) \propto \alpha(H) = 0$, so $\beta(H) = 0$

so $\mathfrak{g}_{\pm\beta} \subset \mathfrak{z}_{\mathfrak{g}}(\mathcal{U}_\alpha)$ so $\mathfrak{g}_{\pm\beta} \subset \mathrm{Lie}(G_\alpha)$

Also disjoint from $\mathcal{U}_\alpha$, so $\bigoplus_{\beta \text{ proportional to } \alpha} \mathfrak{g}_\beta$ injects into

$\mathrm{Lie}(\overline{G}_\alpha) \cong \mathfrak{su}(2) =$ 1d torus + 2 1d root spaces

$\Rightarrow \pm\alpha$ only roots proportional to α, $\mathfrak{g}_\alpha$ is 1d

(2) Have map $N_{G_\alpha}(T) \to N_{\overline{G}_\alpha}(\overline{T})$ $\quad \overline{T} = T/\ker(\chi_\alpha)$

conversely if image $\overline{g} \in \overline{G}_\alpha$ of $g \in G_\alpha$ normalizes $\overline{T}$, have $\forall\, t \in T$ that $\overline{g}\,\overline{t}\,\overline{g}^{-1} \in \overline{T}$, i.e.

$$g t g^{-1} \in T \cdot \ker(\chi_\alpha) = T$$

so g normalizes T, so map is surjective.

if $\overline{g}\,\overline{t}\,\overline{g}^{-1} = \overline{t}$ then $g t g^{-1} = tu$ for some $u \in \exp(\mathfrak{u}_\alpha) = \ker(\chi_\alpha)$

On lie algebra level if $H \in \mathfrak{t}$ have $X \in \mathfrak{u}_\alpha$ st.

$$\mathrm{Ad}_g \cdot H = H + X$$

since $g \in Z_{G_\alpha}(\mathfrak{u}_\alpha)$, $\mathrm{Ad}_g \cdot X = X$

$\Rightarrow$ (induction) $\mathrm{Ad}_{g^n} \cdot H = H + nX$

if $X \neq 0$

$\mathrm{Ad}_g|_{\mathfrak{t}}$ has infinite order. But $W(G_\alpha : T)$ is finite.

Thus $g \in Z_G(\mathfrak{t}) = Z_G(T)$ and the map is an iso.

$$N_{SU(2)}\begin{pmatrix} * & \\ & * \end{pmatrix} = \left\{ \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}, \begin{pmatrix} & -e^{i\theta} \\ e^{i\theta} & \end{pmatrix} \right\}$$

$$= T \cdot \left\{ 1, \underbrace{\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{s_\alpha} \right\}$$

(3) let $s_\alpha \in W(G_\alpha : T)$ be the nontrivial element.

Then s_α fixes u_α pointwise (in centralizer) acts by inversion on $\bar{t} = t/u_\alpha$ $s_\alpha(\bar{H}) = -\bar{H}$

so it's a reflection by u_α in t.

NB

Call a root reduced if it's not a multiple of another root. Here (G cpt) all roots are reduced

$N_{G_\alpha}(T) \subset N_G(T)$ so can think of $s_\alpha \in N_{G_\alpha}(T)/T$ as an element of $N_G(T)/T$.

If we fix a G-inv't inner prdt on $\mathfrak{g}$, s_α acts by isometries so it's the orthogonal reflection by u_α.

Also: W acts on $\mathfrak{t}$ $\rightsquigarrow$ W acts on $\mathfrak{t}^*$,
W preserves Λ $\rightsquigarrow$ W preserves Λ^*.

Also, W preserves $\Phi \subset \Lambda^*$:

If $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{g}_\alpha$, $s \in W$,

$$[s \cdot H, X_\alpha] = \alpha(s \cdot H) \cdot X_\alpha = ({}^t s^{-1} \cdot \alpha)(H) \cdot X_\alpha$$

But $[s \cdot H, X_\alpha] = [H, \mathrm{Ad}(s^{-1}) \cdot X_\alpha]$

$\Rightarrow s^{-1} \cdot \mathfrak{g}_\alpha = \mathfrak{g}_{{}^t s^{-1}\alpha}$.

What is $s_\alpha(\alpha)$? this is a root which vanishes on $\mathfrak{u}_\alpha = \ker(\alpha)$ $\Rightarrow$ proportional to α $\Rightarrow -\alpha$. (modulo $\mathfrak{u}_\alpha$ it's an inversion).

<u>Example:</u> $G = U(n)$, $T = \left\{ \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \right\}$

$\mathfrak{t} = \{ \mathrm{diag}(i\theta_1, \dots, i\theta_n) \}$. $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{gl}_n \mathbb{C}$

<u>roots:</u> $\alpha_{ij}(\theta_1, \dots, \theta_n) = \theta_i - \theta_j$

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \cdot \underbrace{\begin{array}{c|c} & \downarrow j \\ \hline \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix} & \leftarrow i \end{array}}_{E^{ij}} \begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n} \end{pmatrix}$$

$$= e^{i(\theta_i - \theta_j)} E^{ij}.$$

$$u_\alpha = \{ i\,\mathrm{diag}(\theta_1, \dots, \theta_n) \mid \theta_i = \theta_j \}$$

$$\mathrm{Lie}\, \bar{G}_{\alpha,\mathbb{C}} = \mathrm{Span}_{\mathbb{C}} \{ \mathrm{diag}(0, \dots, 0, \underset{i}{1}, 0, \underset{j}{-1}, 0 \cdots 0), \underset{\mathfrak{g}_\alpha}{E^{ij}}, \underset{\mathfrak{g}_{-\alpha}}{E^{ji}} \}$$

$$S_\alpha = \begin{array}{c|c} & \begin{matrix} i & & j \end{matrix} \\ \hline \begin{matrix} i \rightarrow \\ j \rightarrow \end{matrix} & \begin{pmatrix} & & +1 \\ 1 & & \end{pmatrix} \end{array}, \text{ jointly generate } S_n.$$

Def: Call S_α the root reflection associated to $\alpha \in \Phi$. The algebraic Weyl group is the subgp of $W = W(G:T)$ generated by the root reflections

(Thm: this is all of W)

Corollary: Let $\mathfrak{z} = Z(\mathfrak{g})$, $V = \{ v \in \mathfrak{t}^* \mid v(\mathfrak{z}) = 0 \} = (\mathfrak{t}/\mathfrak{z})^*$

Then (V, Φ) is a root system:

(1) $\Phi \subset V$ is finite, does not contain 0

(2) $\mathrm{Span}_{\mathbb{R}} \Phi = V$

(3) For every $\alpha \in \Phi$, the reflection s_α of V in the hyperplane perpendicular to α preserves Φ