Math 535, Lecture 14, 8/2/2023

Last time: view $X \in \mathfrak{g}$ as infinitesimal elements of G. E.g. $\frac{d}{dx}$ is an infinitesimal translation, $\frac{d}{d\theta}$ is an " rotation.

via $L_{\exp(tX)} = \exp(t\,\mathrm{ad}_X)$

(i.e. $f(x+t) = \left(\exp\left(t\frac{d}{dx}\right)\cdot f\right)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\frac{d^kf}{dx^k}(x)$

$f(R(\theta)\cdot \underline{x}) = \left(\exp\left(\theta \frac{d}{d\theta}\right)\right) f(\underline{x})$)

(recall $\left(1+\frac{t}{n}X\right)^n \to \exp(tX)$)

$\Rightarrow$ adjoint rep'n : $\mathrm{Ad}_g(x) = gxg^{-1}$ $(\mathrm{Ad}_g \in \mathrm{Aut}(G))$

$\rightsquigarrow$ $\mathrm{Ad}_g(X) \text{ "=" } gXg^{-1}$

$\in \mathrm{Hom}_{gp}(G, GL(\mathfrak{g}))$

$\rightsquigarrow$ $\mathrm{ad}_X(Y) = [X,Y]$; $\mathrm{ad} \in \mathrm{Hom}_{\text{lin alg}}(\mathfrak{g}, \mathrm{End}_{\text{vsp}}(\mathfrak{g}))$

Prop: $\mathrm{Ad}_g(G) \cong G/Z(G)$ (G ctd)

Today: ① abelian Lie grps
② cpt Lie grps

Cor: Let G be connected. Then TFAE:
(1) $\mathfrak{g}$ is abelian
(2) G is abelian
(3) $\exp: \mathfrak{g} \to G$ is a surjective grp hom

Pf: (1) $\Rightarrow$ (2) If $ad_X = 0$ for all X then $\exp(ad_X) = id$ for X near 0, so $Ad_g \in Aut(\mathfrak{g})$ are 1 for g near 1_G. Since G is ctd it is generated by the nbd of 1_G and $Ad_g \equiv 1$

(2) $\Rightarrow$ (3) If G is abelian, for any $X, Y \in \mathfrak{g}$ $t \mapsto \exp(tX)\exp(tY)$ is a cts grp hom $\mathbb{R} \to G$, hence of the form $\exp(tZ)$.

Diff at $t=0$ shows $Z = X+Y$, so $\exp(tX)\exp(tY) = \exp(t(X+Y))$.
Set $t=1$ to see $\exp \in Hom((\mathfrak{g}, +), (G, \cdot))$

Image is a subgrp contains nbd of 1_G hence is G.

(3) $\Rightarrow$ (1) if $\exp: (\mathfrak{g}, +) \to (G, \cdot)$ is a surjective hom then its image is abelian. $\blacksquare$

<u>Theorem:</u> A connected abelian Lie group has the form $\mathbb{R}^a \times T^b$, its exponential map is a covering map

<u>Pf:</u> G is image of $\exp: \mathfrak{g} \to G$ so

$$G \cong \mathfrak{g} / \ker(\exp).$$

closed subgp of $\mathfrak{g}$

<u>Fact:</u> a closed subgp of $\mathbb{R}^n$ has form $\mathbb{R}^m \oplus \Lambda$ $\quad \Lambda \subset \mathbb{R}^{n-m}$ is a lattice

Here exp is a local diffeo so $\ker(\exp) = \Lambda$ is a discrete subgp of $\mathbb{R}^n$.

<u>Ex:</u> discrete subgp of $\mathbb{R}^n$ has the form

$$\bigoplus_{i=1}^{b} \mathbb{Z} \cdot v_i \quad \text{where } \{v_i\}_{i=1}^{b} \subset \mathbb{R}^n \text{ are linearly indep}$$

$\blacksquare$

Ex: A compact abelian Lie group has the form

$$T^b \times A$$

where A is a finite abelian group.

Part 3: Compact Lie groups

Thm: Let G be a cpt Lie gp. Then G is isomorphic to a closed subgp of $U(n)$ ($\iff$ G has a faithful f.d. rep'n)

Pf: A compact Lie gp has finitely many connected components (G/G° is a discrete cpt gp)

Map $H \mapsto (\dim H, \# \pi_0(H))$

$$\begin{Bmatrix}\text{closed} \\ \text{subgps}\end{Bmatrix} \to \omega \times \omega$$

respects order ($\subseteq$ on left, lexicographic on right)

RHS is a well-ordering $\Rightarrow$ no ∞ descending sequence of closed subgps (compactness is essential: $\mathbb{Z} \supseteq 2\mathbb{Z} \supseteq 4\mathbb{Z} \supseteq 8\mathbb{Z} \supseteq \cdots$)

(no bound on length: in $S^1 \times S^1$
can take $\mathbb{R}/\mathbb{Z} \times \frac{1}{k}\mathbb{Z}/\mathbb{Z} \supset \frac{1}{m}\mathbb{Z}/\mathbb{Z} \supset \ldots$)

The representation of G on $L^2(G)$ is faithful
From Peter-Weyl:

$$\bigcap_{\pi \in \hat{G}} \ker(\pi) = \{e\}$$

Enumerate rep'ns $\pi_1, \pi_2, \ldots$ define

$$H_n = \bigcap_{i=1}^{n} \ker(\pi_i)$$

descending sequence of closed subgroups, so stabilizes

Say $H_n = H_N$ for all $n \geq N$.

But then $H_N = \{e\}$, so $\bigoplus_{i=1}^{N} \pi_i$ is faithful.

Remark: In fact image is algebraic (Zariski-closed) !!!

Tool for structure theory: tori

Key step: $\mathrm{Hom}(\mathbb{T}^n, \mathbb{T}^m)$ $\quad (\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n)$

Let $f: \mathbb{Z}^n \to \mathbb{Z}^m$ be a gp hom

Extending scalars gives a linear map $f_{\mathbb{R}}: \mathbb{R}^n \to \mathbb{R}^m$
$f_{\mathbb{R}} = 1_{\mathbb{R}} \otimes_{\mathbb{Z}} f.$

Now $f_{\mathbb{R}}(\mathbb{Z}^n) \subseteq \mathbb{Z}^m$ so induces a map

$$\tilde{f}: \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^m/\mathbb{Z}^m.$$

Lemma: The map $f \mapsto \tilde{f}$ is an isom

$$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m) \to \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m).$$

Proof: For inverse, let $\exp_n: \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the exponential map = quotient map.

Given $\tilde{f} \in \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$,
$d\tilde{f}: \mathbb{R}^n \to \mathbb{R}^m$ is linear

has: $\exp_m(d\hat{f}) = \hat{f} \circ \exp_n$

evaluating at $\underline{x} \in \mathbb{Z}^n$ get $d\hat{f}(\underline{x}) \in \mathbb{Z}^m$.

so $d\hat{f} = f_{\mathbb{R}}$ for some $f: \mathbb{Z}^n \to \mathbb{Z}^m$.

<u>Cor:</u> $\mathrm{Aut}(\mathbb{T}^n) \cong M_n(\mathbb{Z})^\times = GL_n(\mathbb{Z})$

($\mathrm{Aut}(\mathbb{R}^n/\mathbb{Z}^n)$ is discrete!)

<u>Cor:</u> $\hat{\mathbb{T}}^n = \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}^n$

homs $\mathbb{T}^n \to S^1$ are $\underline{x} + \mathbb{Z}^n \mapsto e(\underline{k} \cdot \underline{x})$
where $\underline{k} \in (\mathbb{Z}^n)^\vee$
$= \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$.

(usually write
$e(z) = e^{2\pi i z}$)