Math 535, lecture 12, 3/2/2023

Last time: Exponential map

Def: $e_X : \mathbb{R} \to G$ the integral curve of $X \in \mathfrak{g}$ s.t. $e_X(0) = e_G$.

Fact: lives forever, $e_X(t+s) = e_X(t) \cdot e_X(s)$, depends only on tX, $\Rightarrow$ write $\exp(X) = e_X(1)$

Fact: $\exp : \mathfrak{g} \to G$ a local diffeo, $d_0 \exp = \mathrm{id}_{\mathfrak{g}}$.

$\Rightarrow$ Exponential co-ordinates: $\mathfrak{g} = \bigoplus_{i=1}^{r} V_i \ni (X_i)_{i=1}^{n} \mapsto \prod_{i=1}^{r} \exp(X_i) \in G$.

Fact: $f \in \mathrm{Hom}(G, H)$ has $f(\exp_G(X)) = \exp_H(df(X))$

Today: Closed subgroups

G Lie gp, $H \subset G$ closed subgp. Goal: H is a Lie subgp

Key: identify $\mathrm{Lie}\, H \subset \mathrm{Lie}\, G$.

Fixed $0 \in \mathcal{U}_{\mathfrak{g}} \subset \mathfrak{g}$, $\mathcal{U} \subset G$ open s.t. $\exp : \mathcal{U}_{\mathfrak{g}} \to \mathcal{U}$ is a diffeomorphism onto. $\log : \mathcal{U} \to \mathcal{U}_{\mathfrak{g}}$ inverse

$$\mathfrak{h}_1 = \mathbb{R} \cdot \left\{ \lim \frac{\log h_n}{|\log h_n|} \;\middle|\; \begin{matrix} (h_n)_{n \geq 1}^{\infty} \subset H \\ h_n \to e \end{matrix} \right\}$$, $|\cdot|$ on $\mathfrak{g}$ any norm.

$$\mathfrak{h}_2 = \{ X \in \mathfrak{g} \mid \exp(tX) \in H \text{ for all } t \}$$

$$\mathfrak{h}_3 = \mathbb{R} \cdot \log(H \cap \mathcal{U}).$$

if $X \in \mathfrak{h}_2$ then for $t_n \to 0$, $\log(\exp(t_n X)) = t_n X$

and $\frac{t_n X}{|t_n X|} \to \frac{X}{|X|}$ so $X \in \mathfrak{h}_1$

Similarly if t small, $\exp(tX) \in \mathcal{U}$ so $tX \in \log(H \cap \mathcal{U})$ so $X \in \mathfrak{h}_3$

<u>Converse:</u> $\mathfrak{h}_1 \subset \mathfrak{h}_2$. <u>Pf:</u> Let $\frac{\log h_n}{|\log h_n|} \to X \in \mathfrak{g}$.

Then for $m_n \in \mathbb{Z}$, $H \ni h_n^{m_n} = \exp(m_n \cdot \log h_n) =$

$$= \exp\left(\frac{\log h_n}{|\log h_n|} \cdot (m_n \cdot |\log h_n|) \right)$$

since $h_n \to e$, $|\log h_n| \to 0$, can choose m_n s.t. $m_n \cdot |\log h_n| \underset{n\to\infty}{\to} t$ then

$$h_n^{m_n} \to \exp(tX)$$

But H is <u>closed</u>, so $\exp(tX) \in H$ for all t, and $X \in \mathfrak{h}_2$.

<u>Goal:</u> $\log(H \cap U) \subset \mathfrak{h}_1 = \mathfrak{h}_2$

<u>Claim:</u> for a small enough nbd $e \in U_1 \subset U$, $\log(H \cap U_1) \subset \mathfrak{h}_1 = \mathfrak{h}_2$.

<u>Pf:</u> If not, we have $\{h_n\}_{n=1}^{\infty} \subset H$ s.t. $h_n \to e$, $\log h_n \notin \mathfrak{h}_1 = \mathfrak{h}_2$.

Fix a subspace $\mathfrak{k} \subset \mathfrak{g}$ s.t. $\mathfrak{h}_2 \oplus \mathfrak{k} = \mathfrak{g}$

then have $h_n = \exp(X_n)\exp(Y_n)$ for some $X_n \in \mathfrak{h}_1$, $Y_n \in \mathfrak{k}$ (exponential co-ords) with $X_n \to 0$, $Y_n \to 0$.

Then $\exp(Y_n) = \exp(-X_n) h_n \in H$ tend to e

So if $Y_n \neq 0$, any subsequential limit of $\frac{Y_n}{|Y_n|}$ lies in $\mathfrak{h}_1$; But it also lies in k since $Y_n \in k$

So $Y_n = 0$ from some point on, $h_n \in \exp(\mathfrak{h}_1)$

$\Rightarrow$ log identifies $H \cap U_1$ with $\mathfrak{h}_1 \cap \log(U_1)$

This shows H is a submanifold of G near e, by translation-invariance of H, H is a submanifold.

Payback; $\mathfrak{h}_1 = \mathfrak{h}_2$ is a subspace. Need to show closed under $+$: Say $X, Y \in \mathfrak{h}_1 = \mathfrak{h}_2$
Then

$$\exp_G(tX) \exp(tY) \in H$$

we know $d_0(\exp_G X \cdot \exp_G Y) = X + Y$
$\mathfrak{g} \times \mathfrak{g} \to G$

so $\log(\exp_G(tX) \exp_G(tY)) = t(X+Y) + O(t^2)$

so $\dfrac{\log(\qquad\qquad)}{|\log(\qquad\qquad)|} = \dfrac{X+Y}{|X+Y|} + O(t)$

$$\downarrow$$

$\dfrac{X+Y}{|X+Y|}$ as $t \to 0$

so $X+Y \in \mathfrak{h}_1 = \mathfrak{h}_2$. $\blacksquare$

<u>Cor</u>: Let $f: G \to H$ be a <u>cts</u> gp hom of Lie groups. Then f is smooth.

<u>Pf</u>: $\Gamma_f = \{(g, f(g))\} \subset G \times H$ is a closed subgp. $\blacksquare$

<u>Topological applications</u>

<u>Thm</u>: Let $H < G$ be closed, ctd. Then G/H has a unique manifold structure st $\pi: G \to G/H$ is smooth. The regular action is smooth.

<u>Pf</u>: H is closed $\Leftrightarrow$ G/H is Hausdorff in quotient topology. To get manifold structure fix complement $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$

Then $\exp|_{\mathfrak{k}}$ is a co-ord system on G/H near identity.

∎

(if H normal, this makes G/H a Lie gp)

<u>Thm:</u> A Lie gp hom $f: G \to H$ is a covering map iff df is an isomorphism (assume G, H ctd)

<u>Pf:</u> ① a cover is a local diffeo, so df is an isomorphism

② Suppose d_ef is an isom, so d_gf is an isom $\Rightarrow f$ is a local diffeomorphism

Let $K = \operatorname{Ker} f$. This is a closed subgp, with Lie algebra $\operatorname{Ker} df = \{0\}$ so K is 0-dim, i.e. discrete. So let U be small enough s.t. K-translates of U are disjoint, s.t. $f|_U$ is a diffeo. Then

$$f^{-1}(f(U)) = K \times U.$$

∎

Thm: let $df : \mathfrak{g} \to \mathfrak{h}$ be a map of Lie algebras. $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{h} = \mathrm{Lie}(H)$.
suppose H cts, G ctd & simply ctd. Then df lifts to gp hom $G \to H$

Pf: let $\Gamma_{df} = \{(X, df \cdot X)\} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of df, a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$

(image of $id \oplus df : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{h}$). Let Γ_f be the corresponding subgp of $G \times H$.

Have projections
$$\begin{tikzcd} & G \times H \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G & & H \end{tikzcd}$$
are homs

Let $\pi = \pi_1|_{\Gamma_f}$. $d\pi = d\pi_1|_{\Gamma_{df}} : \Gamma_{df} \to \mathfrak{g}$ is projection on 1st co-ord, an isom.

$\Rightarrow$ map $\pi : \Gamma_f \to G$ is a covering map

By hypothesis (G simply ctd) π is an isom

This means $\pi^{-1} : G \to \Gamma_f$ is an isom, and $\pi_2 \circ \pi_1^{-1}$ is a Lie gp hom with graph Γ_f, hence derivative df

$$\begin{array}{ccc} \mathfrak{g} \oplus \mathfrak{h} & & G \times H \\ \cup & & \cup \\ \Gamma_{df} & & \Gamma_f \end{array}$$

∎

(Ex: if $\Gamma \subset G \times H$ is a subgp s.t. $\pi_1|_\Gamma$ is an isom $\Gamma \to G$ then Γ is the graph of a gp hom)

<u>Thm</u>: (Ado) Every f.d. Lie algebra has a faithful linear representation (embedding in $\mathfrak{gl}_n\mathbb{R}$)

<u>Cor</u>: Every f.d. Lie algebra is the Lie algebra of some Lie gp (subgp of $GL_n(\mathbb{R})$).

<u>Example</u>: $\pi_1(SL_2(\mathbb{R})) \cong \mathbb{Z}$

But if G covers $SL_2(\mathbb{R})$, any hom $G \to GL_n(\mathbb{R})$ factors through $SL_2(\mathbb{R})$