Math 535, lecture 10, 30/1/2023

Last time: ① $f: M \to N \rightsquigarrow df_p : T_pM \to T_pN$.

② Curves tangent to vector fields.
$\Downarrow$
submanifolds tangent to distributions
(if $[V, V] \subseteq V$)

Ex: let $f: M \to \mathbb{R}$ ① $df_p \in \operatorname{Hom}(T_pM, T_{f(p)}\mathbb{R})$
$= (T_pM)' = T_p^*M$

② $df_p = (f - f(p)) + I_p^2 \in I_p / I_p^2$

Today: Lie groups!

Def: A **Lie group** is a group object in the category of smooth manifolds, i.e. a smooth manifold G equipped with smooth map $\cdot : G \times G \to G$ and $\cdot^{-1} : G \to G$ s.t. $(G, \cdot, ^{-1})$ is a group

A **homomorphism** of Lie groups is a smooth map which is also a group hom

<u>Fact:</u> A cts / measurable gp hom of Lie gps is smooth

<u>Ex:</u> Do this for maps $(\mathbb{R},+) \to (\mathbb{R},+)$

<u>Examples:</u> $(\mathbb{R}^n,+)$, $(\mathbb{R}^n/\mathbb{Z}^n,+) \cong (\mathbb{R}/\mathbb{Z})^n$

<u>Example:</u> Discrete gps (0-dim Lie gps)

<u>Example:</u> $GL_n(\mathbb{R}) \underset{\text{open}}{\subset} M_n(\mathbb{R})$, $GL_n(\mathbb{C})$

$SL_n(\mathbb{R}) = \{ g \in M_n(\mathbb{R}) \mid \det g = 1 \}$

$O(p,q) = \{ g \in GL_{p+q}(\mathbb{R}) \mid {}^t g \, I_{p,q} \, g = I_{p,q} \}$

$I_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ $\vdots$

<u>Example:</u> If G, H are Lie groups, so is $G \times H$, semidirect products.

<u>Example:</u> $\mathbb{E}^n = (\mathbb{R}^n, \|\cdot\|_2)$: $\mathrm{Isom}(\mathbb{E}^n) = O(n) \ltimes \mathbb{R}^n$

$\subseteq \mathrm{Aff}_n(\mathbb{R}) = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$

Example: (M,g) Riemannian manifold
$\rightsquigarrow$ $\mathrm{Isom}(M,g)$ is a Lie group.

Leads to ∞-dim generalizations such as
$\mathrm{Diffeo}(M)$,

Def: An action of a Lie group G on a manifold M is a smooth map $\cdot : G \times M \to M$ which is also a group action

Def: A Lie subgroup of a Lie group is a submanifold which is also a subgroup
$\Leftrightarrow$ image of an injective hom of Lie groups

Example: ① subgps of $\mathbb{R}^2$ are the linear subspaces

② subgps of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ are images of those
If $L \subset \mathbb{R}^2$ is a line, image of L mod $\mathbb{Z}^2$
is $<$ closed if L has rational slope
dense otherwise

Remark: Sophus Lie studied local Lie groups

Lie algebras

G acts on itself by left multiplication (the "regular action"), hence on $\mathcal{D}_G$

Call $X \in \mathcal{D}_G$ "left-invariant" if it's fixed by this action: $g \cdot X = X$.

Defi: The Lie algebra of G is the set

$$\operatorname{Lie} G = \mathfrak{g} = \left\{ \begin{array}{l} \text{left-inv't vector} \\ \text{fields on } G \end{array} \right\}.$$

Lemma: This is a Lie subalgebra of $\mathcal{D}_G$.
Pf: $g \cdot [X, Y] = [gX, gY]$.

Recall have a surjective restriction map
$$\mathcal{D}_G \to T_e G$$

Lemma: Restricting this map to $\operatorname{Lie} G$ gives a linear isom.
Pf: Need the inverse. For any G-manifold M, action extends to TM by $g \cdot (p, v) = (gp, dL_g \cdot v)$

Here, if $v \in T_e G$, the orbit $g \mapsto (g, dL_g \cdot v)$ is a smooth vector field.

<u>Thm:</u> If $f \in \mathrm{Hom}(G, H)$ then $df \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$

as Lie gps — as Lie algebras

<u>Pf:</u> If $\varphi \in C^\infty(H)$, then

$$(df([X,Y]) \cdot \varphi)(e_H) \underset{\substack{\uparrow \\ \text{defn of } df_e,\ f(e_G)=e_H}}{=} ([X,Y] \cdot (\varphi \circ f))(e_G)$$

$$= (X Y (\varphi \circ f) - Y \cdot X \cdot \varphi \circ f)(e)$$

$$\overset{*}{=} df_e(X)\, df_e(Y) \cdot \varphi - df_e(Y)\, df_e(X) \cdot \varphi$$

$$= ([df(X), df(Y)] \cdot \varphi)(e_H)$$

∎

Remark: Converse TBD.

($\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ local diffeo., no inverse)

<u>Lemma:</u> If H is a ctd top gp, $U \subset H$ open, then $\langle U \rangle = H$.

Pf: $L = \langle U \rangle$ is an abstract subgp. Also

$$L = LU = \bigcup_{h \in L} hU \subset H \text{ is open}$$

$\Rightarrow L \subset H$ is closed ($H \setminus L$ is a union of L-cosets)

$\Rightarrow L = H$ (H is ctd)

Thm: Every Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$ is the subalgebra of a Lie subgp

Pf: The span of the vector fields in $\mathfrak{h}$ is a distribution on G. (set $V_g = g \cdot \mathfrak{h}$)
The distribution is integrable by defn of Lie subalgebra. By Frobenius, have submanifold tangent to V_g through each pt.

Let H be the leaf through e. This leaf is H-inv't, hence a subgp

(whole foliation is left-G-inv't)
if $h \in H$, $h \cdot H \ni h \cdot e = h$ so $h \cdot H = H$

thus also $1 \in hH$ so $h^{-1} \in H$.. $\blacksquare$