Math 535, lecture 3, 13/1/2023

Last time: (1) topological groups, subgroups
(2) continuous representations.

Today: Representation theory.

Representation: map $\pi : G \to GL(V)$ s.t.
$G \times V \ni (g, \underline{v}) \mapsto \pi(g)\underline{v} \in V$
is cts

Let (π, V), (σ, W) be representations of G.
Then the following are representations:

(1) $(\check{\pi}, V')$ contragradient representation:
V' = top. dual, $\check{\pi}(g) = (\pi(g)^{-1})'$

(If V is a Banach space equip V' with the norm topology; $\| \pi(g)' \|_{V'} = \| \pi(g) \|_V$)

(2) direct sum $(\pi \oplus \sigma, V \oplus W)$
← pdt topology.

(3) quotient If $\mathcal{U} \subset V$ closed, G-inv't def

$$\bar{\pi}(g)(\underline{v} + \mathcal{U}) = \pi(g)\underline{v} + \mathcal{U}$$

This gives a rep'n on $V/\mathcal{U}$ (quotient topology)

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(4) If (σ, W) is a rep'n of H, then $G \times H$ acts linearly on tensor product $V \otimes W$ by

$$[(\pi \otimes \sigma)(g,h)] \cdot (\underline{v}, \underline{w}) = (\pi(g)\underline{v}) \otimes (\sigma(h)\underline{w}).$$

If V, W are f.d. unique topology on $V \otimes W$ (it's f.d. too) gen rep'n $\pi \otimes \sigma$.

In ∞-dim case: different completions, theory of tensor pdts (c.f. Grothendieck)

Ex: If $(\pi, V) \in \mathrm{Rep}(G)$, $\mathcal{U} \subset V$ G-inv't subspace so is $\bar{\mathcal{U}}$.

Def: subrep'n = closed inv't subspace of V.

Call (π, V) irreducible if only subrep'ns are $\{0\}, V$.

Example: Let $\mathbb{R}$ act on $\mathbb{R}^2$ by

$$\pi(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

then $\mathbb{R}\cdot\begin{pmatrix}1\\0\end{pmatrix}$ is invt. No invt complement
$\therefore$ rep'n is reducible but not decomposable.

Matrix coefficients

If $\pi: G \to GL_n(\mathbb{C})$, what is $\pi(g)_{ij}$?

these are the coeffs of $\pi(g)\underline{e}_j$ wrt $\{\underline{e}_i\}_{i=1}^{n}$

i.e. $\langle \underline{e}_i', \pi(g)\underline{e}_j \rangle$.

Def: $(\pi, V) \in \mathrm{Rep}(G)$. A matrix coefficient of V is a function

$$\Phi_{\underline{v},\underline{v}'}(g) = \langle \underline{v}', \pi(g)\underline{v} \rangle$$

where $\underline{v} \in V$, $\underline{v}' \in V'$.

Clear: π cts $\Rightarrow \Phi_{\underline{v},\underline{v}'} \in C(G)$ (cts fcn on G)

Remark: Often care about both smoothness and decay of matrix coeff.

Lemma: map $V \times V' \to C(G)$ $(\underline{v}, \underline{v}') \mapsto \Phi_{v,v'}$ is bilinear, extends to an intertwining operator of algebraic $G \times G$-rep'ns

$$(\pi \otimes \check{\pi}, V \otimes V') \longrightarrow C(G)$$

$$((g_1, g_2) \cdot f)(x) = f(g_2^{-1} x g_1).$$

Pf: $\Phi_{\pi(g_1)\underline{v}, \check{\pi}(g_2)\underline{v}'}(x) = \langle \check{\pi}(g_2)\underline{v}', \pi(x) \cdot \pi(g_1)\underline{v} \rangle$

$= \langle \pi(g_2^{-1})' \underline{v}', \pi(x g_1) \underline{v} \rangle = \langle \underline{v}', \pi(g_2^{-1}) \pi(x g_1) \underline{v} \rangle$

$= \langle \underline{v}', \pi(g_2^{-1} x g_1) \underline{v} \rangle = \Phi_{\underline{v}, \underline{v}'}(g_2^{-1} x g_1)$

$= ((g_1, g_2) \cdot \Phi_{v, \underline{v}'})(x).$ ∎

Cor: Can realize every rep'n of G on a subspace of $C(G)$ (might not be closed!)

Observations: (1) if V is f.d. so is $V \otimes V'$ and image is closed

(2) If G is cpt, $\Phi_{v,v'}$ is bdd & square-integrable

Def: Say that an irrep (π, V) of G (assume loc. cpt) belongs to the discrete series if it's isomorphic to a subrep'n of $L^2(G)$ (wrt Haar measure)

Fact: A locally cpt top sp G has a measure μ which is left-inv't: $\mu(gA) = \mu(A)$. Called Haar measure, unique up to rescaling.

Observe: If V_π Hilbert space, π unitary then $V_\pi' \cong V_\pi$, so

$$|\Phi_{\underline{v}, \underline{v}'}(g)| \leq \|\underline{v}'\| \cdot \|\underline{v}\|$$

Def: $\Phi_{\sum_i \underline{v}_i' \otimes \underline{v}_i}(g) = \sum_i \Phi_{\underline{v}_i', \underline{v}_i}(g)$

$$L^p(X, \mu) = \{ f : X \to \mathbb{C} \mid \int |f|^p d\mu < \infty \} / \{ f \mid f = 0 \text{ a.e.} \}$$

with norm $\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$

Ex: on $GL_n(\mathbb{R})$, Haar measure has density $\frac{1}{|\det g|^n}$ wrt Lebesgue on $M_n(\mathbb{R})$