Lior Silberman's Math 535, Problem Set 6: Preliminaries on Tori

Connected abelian Lie groups

1. Let $\Lambda < \mathbb{R}^d$ be a discrete subgroup. Show that $\Lambda = \oplus_{i=1}^k \mathbb{Z}\underline{v}_i$ for a linearly independent set $\{\underline{v}_i\}_{i=1}^k \subset \mathbb{R}^d$. Conversely show that such a subgroup is discrete.
2. Let G be an Abelian Lie group, and suppose that $\pi_0(G) = G/G^\circ$ is finite. Show that $G \simeq G^\circ \times \pi_0(G)$. (Hint: show that a connected abelian Lie group is divisible).

Tori

3. (Fourier analysis on tori) Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus. A *trigonometric polynomial* on $\mathbb{T}^n$ is a function of the form $f(\underline{x}) = \sum_{i=1}^I a_i e(\underline{k}_i \cdot \underline{x})$ where $\underline{k}_i \in (\mathbb{Z}^n)^*$ lie in the dual lattice.
   (a) Use Peter–Weyl to show that the space of trigonometric polynoimals is dense in $C(\mathbb{T}^n)$ and $L^2(\mathbb{T}^n)$.
   (b) Use Stone–Weierstrass instead to show that the trigonometric polynomials are dense in $C(\mathbb{T}^n)$, and use that to show that their orthocomplement in $L^2(\mathbb{T}^n)$ vanishes, getting density there too.
   (c) For $f \in L^2(\mathbb{T}^n)$ and $\underline{k} \in (\mathbb{Z}^n)^*$ set $\hat{f}(\underline{k}) = \int_{\mathbb{T}^n} f(\underline{x}) e(-\underline{k} \cdot \underline{x})\, d^n x$ (probability Haar measure). Then $\sum_{\underline{k}} \hat{f}(\underline{k}) e(\underline{k} \cdot \underline{x})$ converges in L^2 to f.
   (d) For $f \in C^m(\mathbb{T}^n)$ use integration by parts to show that $\left|\hat{f}(\underline{k})\right| \le C_f (1 + |\underline{k}|)^{-m}$. Conclude that for $m > n$, the series $\sum_{\underline{k}} \hat{f}(\underline{k}) e(\underline{k} \cdot \underline{x})$ converges in C^{m-n-1} to f.
   (e) (Weyl criterion) Let $\{\mu_j\}_{j=1}^\infty$ be a sequence of Borel probability measures on $\mathbb{T}^n$. Show that $\mu_j(f) \to \mu(f)$ for every f iff this holds for the *plane waves* $f(\underline{x}) = e(\underline{k} \cdot \underline{x})$
4. (Weyl equidistribution) Let $\{\xi_i\}_{i=0}^n \subset \mathbb{R}$ be linearly independent over $\mathbb{Q}$ where $\theta_0 = 1$, and let $\underline{\xi} = (\xi_i)_{i=1}^n \mod \mathbb{Z}^n \in \mathbb{T}^n$. Show that the sequence $\left\{k\underline{\xi}\right\}_{k=1}^\infty \subset \mathbb{T}^n$ is *uniformly distributed*: for any open $U \subset \mathbb{T}^n$,

$$\frac{1}{K}\#\left\{1 \le k \le K \mid k\underline{\xi} \in U\right\} = \frac{\mathrm{vol}(U)}{\mathrm{vol}(\mathbb{T}^n)}.$$

   Conclude that the sequence $\left\{k\underline{\xi}\right\}_{k=1}^\infty$ is *dense* in the torus.

   Hint: Let $\mu_K = \frac{1}{K}\sum_{k=1}^K \delta_{k\underline{\xi}}$. By 3(e) to show $\mu_K \xrightarrow[K\to\infty]{\text{wk-*}} \mathrm{vol}$ it suffices to test against plane waves.