Lior Silberman's Math 535, Problem Set 5: Lie Groups

Constructions on manifolds

The first two exercises are highly recommended if you are interested in algebraic geometry or in differential geometry:

1. (View of the tangent space) Let M be a smooth manifold, $C^\infty(M)$ its algebra of smooth functions (multiplication defined pointwise). For $p \in M$ let $I_p = \{f \in C^\infty(M) \mid f(p) = 0\}$ be the associated maximal ideal. Recall that we set $T_p^*M = I_p/I_p^2$ and $T_pM = \left(T_p^*M\right)^*$.
   (a) Let G_p be the set of pairs (f,U) where $p \in U \subset M$ is open and $f \in C^\infty(U)$. Show that $(f,U) \sim (g,V) \iff \exists W : f\restriction_W = g\restriction_W$ (here $W \subset U \cap V$ is an open set containing p) is an equivalence relation, and endow $\mathcal{G}_p \stackrel{\text{def}}{=} G_p/\sim$ with a natural structure as an $\mathbb{R}$-algebra.
   (b) Let $C^\infty(M)_{I_p}$ be the *localization* of $C^\infty(M)$ at the prime ideal I_p. Show that associating to $f \in C^\infty(M)$ the equivalence class of $(f,M) \in G_p$ is an algebra homomorphism $C^\infty(M) \to \mathcal{G}_p$ inducing an isomorphism $C^\infty(M)_{I_p} \simeq \mathcal{G}_p$.
   (c) Conclude that restriction of maps induces an isomorphism $I_p(M)/I_p^2(M) \simeq I_p(U)/I_p^2(U)$ for any open U containing p.
   (c) A *derivation* at p is an $\mathbb{R}$-linear map $X\colon C^\infty(M) \to \mathbb{R}$ such that $X(fg) = (Xf)\cdot g(p) + f(p)\cdot(Xg)$. Write $\tilde{T}_pM$ for the set of derivations at p. Show that $\tilde{T}_pM$ is an $\mathbb{R}$-vector space.
   (d) Let $X \in \tilde{T}_pM$. Show that $X(f) = 0$ if $f \in I_p^2$, so that the map $X \mapsto X\restriction_{I_p/I_p^2}$ gives a linear map $\tilde{T}_pM \to T_pM$.
   (e) Conversely, let $v \in T_pM$. Show that setting $X_vf \stackrel{\text{def}}{=} v(f - f(p))$ gives $X_v \in \tilde{T}_pM$ and that the map $v \mapsto X_v$ is inverse to the map of (d).

2. Let M,N be smooth manifolds and let $\varphi\colon M \to N$ be a smooth map. Fix $p \in M$.
   (a) Show that mapping $f \in I_{\varphi(p)}N$ to $f \circ \varphi \in I_p(M)$ induces a linear map $(d\varphi_p)^*: T^*_{\varphi(p)}N \to T_p^*M$.
   (b) For $X \in \tilde{T}_pM$ and $f \in C^\infty(N)$ set $d\varphi_p(X)f \stackrel{\text{def}}{=} X(f\circ\varphi)$. Show that $d\varphi_p(X) \in \tilde{T}_pN$ and that $d\varphi_p \in \mathrm{Hom}_{\mathbb{R}}\left(\tilde{T}_pM, \tilde{T}_pN\right)$.
   (c) Show that, under the isomorphism T_p and $\tilde{T}_p$ from problem 1, the maps $d\varphi_p$ and $(d\varphi_p)^*$ are indeed dual.
   (d) Show that the map $\varphi \mapsto d\varphi$ satisfies the *chain rule*: if $\psi\colon L \to M$ is smooth and $p \in L$ then $d(\varphi\circ\psi)_p = d\varphi_{\psi(p)} \circ d\psi_p$.
   (e) Show that $d\varphi_p, d\varphi_p^*$ extend to *bundle maps* $d\varphi\colon TM \to TN$, $d\varphi^*\colon T^*N \to T^*M$.

3. Let $(M,\mathcal{F})$ be a σ-compact topological space equipped with a sheaf $\mathcal{F}$. Suppose that every $x \in M$ has a neighbourhood U so that $(U, \mathcal{F}\restriction_U)$ is isomorphic to the sheaf of smooth functions on an open set in some $\mathbb{R}^n$. Show that there is a unique manifold structure on M compatible with the sheaf $\mathcal{F}$.

The following two exercises are merely a technical verification.

DEFINITION. Let $\Omega \subset \mathbb{R}^n$ be a domain, V a topological vector space. For $1 \le i \le n$, $f\colon \Omega \to V$ and $x \in \mathbb{R}^n$ set

$$(\partial_i f)(x) = \lim_{h \to 0} \frac{f(x+he_i) - f(x)}{h}$$

(e_i is the unit vector in direction i) provided the limit exists. Write $C^0(\Omega; V) = C(\Omega; V)$ for the space of continuous functions $\Omega \to V$ and then let

$$C^{k+1}(\Omega; V) = \left\{ f \in C^k(\Omega; V) \mid \forall i : \partial_i f \in C^k(\Omega; V) \right\}$$

$$C^\infty(\Omega; V) = \bigcap_{k=0}^{\infty} C^k(\Omega; V).$$

Finally, if V is a normed space and $K \subset \Omega$ is compact we set $\|f\|_{C^k,K} = \sup\{\|\partial^\alpha f(x)\| \mid x \in K, |\alpha| \le k\}$.

3. Show that this definition is independent of the choice of co-ordinates: if $\varphi\colon \Omega \to \Omega'$ is a diffeomorphism then $f \mapsto f \circ \varphi$ is a bijection $C^k(\Omega'; V) \to C^k(\Omega; V)$. In particular, $f \in C^1(\Omega; V)$ has directional derivatives in all directions.

4. Let M be a smooth manifold. Define the spaces $C^k(M; V)$ and $C^\infty(M; V)$. Show that when M is compact the topology of $C^k(M; V)$ is determined by a norm such that $\|f\|_{C^k,M} < \infty$ for all $f \in C^k(M; V)$.

Representation Theory

Fix a Lie group G and a representation $(\pi, V) \in \mathrm{Rep}(G)$.

5. Call $\underline{v} \in V$ *smooth* if the orbit function $g \mapsto \pi(g)\underline{v}$ is a smooth function $G \to V$ in the sense of problem 4. Write V^∞ for the set of smooth vectors in V.
   (a) Show that $V^\infty$is a G-invariant subspace of V.
   (b) Show that V^∞ is *dense* in V (hint: revisit arguments used in the proof of the Peter–Weyl Theorem).
   (c) Suppose V is finite dimensional. Show that every vector in V is smooth (hint: $\pi\colon G \to \mathrm{GL}(V)$ is a continuous homomorphism of Lie groups).

6. For $X \in \mathfrak{g}$ and $\underline{v} \in V^\infty$ set $\pi(X)v = \frac{d}{dt} \restriction_{t=0} \pi\left(e^{tX}\right)\underline{v}$.
   (a) Show that this is *well-defined* (that the derivative above exists) and that $\pi(X)v \in V^\infty$. In fact, show that $\pi(X)\colon V^\infty \to V^\infty$ is linear.
   (b) (Compatibility) Show that $\pi(g)\pi(X)\pi(g^{-1}) = \pi(\mathrm{Ad}_g X)$ for all $g \in G$.
   (c) Show that $X \mapsto \pi(X)$ is a linear map $\mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V^\infty)$.
   (d) Show that we obtained a *Lie algebra representation*: $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X) = [\pi(X), \pi(Y)]$. Here, the first commutator is the one in $\mathfrak{g}$, the second the one of $\mathrm{End}_{\mathbb{C}}(V^\infty)$.

Structure theory

7. Show that $\exp\colon {}_2\mathbb{R} \to \mathrm{SL}_{\not\models}(\mathbb{R})$ is not surjective.

8. Let G be a lie group. Show that the connected component G° (see PS1 problem 1) is open. Conclude that the component group $\pi_0(G)$ is discrete.