Lior Silberman's Math 535, Problem Set 3: Representation Theory

Basic constructions

1. In each case give a precise definition and show that the reuslt is a continuous representation of the appropriate groups. In all cases $(\pi, V) \in \mathrm{Rep}(G)$.
   (a) $H \subset G$ a subgroup, $\mathrm{Res}^G_H \pi$ the *restriction* of π to H (still acting on V).
   (b) $W \subset V$ a closed invariant subspace, $\pi \restriction_W$ the restriction of π to W (still a representation of G).
   (c) $W \subset V$ a closed invariant subspace, $\bar{\pi}$ the representation of G on V/W.
   (d) The representation $\pi \oplus \sigma$ of G on $V \oplus W$ where $(\sigma, W) \in \mathrm{Rep}(G)$ also.

2. Consider the representation $\check{\pi}$ of G on V' where $\check{\pi}(g)\varphi = \varphi \circ \pi(g^{-1})$.
   (a) Show that $\check{\pi}(g) \colon V' \to V'$ is linear and that $\check{\pi}(gh) = \check{\pi}(g)\check{\pi}(h)$.
   (b) Show that $\check{\pi} \colon G \times V' \to V'$ is continuous where V' is equipped with the weak-* topology (the locally convex topology determined by the seminorms $|\varphi|_{\underline{v}} = |\varphi(\underline{v})|$ where $\underline{v} \in V$
   (c) Show that $\check{\pi} \colon G \times V' \to V'$ is continuous where V' is equipped with the *strong topology* (the locally convex topology determined by the seminorms $|\varphi|_E = \sup_{\underline{v} \in E} |\varphi(\underline{v})|$ where E ranges over the bounded subsets of V).
   RMK If V is a Banach space, the strong topology on V' is exactly the norm topology with respect to the dual norm.

3. Let $(\sigma, W) \in \mathrm{Rep}(G)$
   (a) Show that the natural action $\pi \boxtimes \sigma$ of $G \times H$ on the algebraic tensor product $V \otimes W$ defines an action by linear maps.
   (b) Show that this action is a continuous representation if V, W are finite dimensional.

Constructions and Characters

Let $(\pi, V), (\sigma, W) \in \mathrm{Rep}(G)$ be finite-dimensional with characters χ_π, χ_σ..

4. We compute some characters.
   (a) Compute the characters of $\pi \oplus \sigma$, $\pi \otimes \sigma$ in terms of χ_π, χ_σ.
   (b) Let $U \subset V$ be G-invariant, and let $\tau(g) = \pi(g) \restriction_U$. Show that $\chi_{V/U} = \chi_\pi - \chi_\tau$.
   (c) Suppose instead that σ was a representation of a group H and compute the character of $\pi \boxtimes \sigma$ as a function on $G \times H$.

5. (Symmetric and antisymmetric tensor powers)
   (a) For $k \geq 2$ show that $\mathrm{Sym}^k V$, $\bigwedge^k V$ are G-invariant subspaces of $V^{\otimes k}$.
   DEF Write $\mathrm{Sym}^k \pi$, $\bigwedge^k \pi$ for the resulting representations.
   (b) Find the character of $\mathrm{Sym}^k \pi$

Examples of characters

6. Let G be a finite group and let X be a finite G-set.
   (a) Show that setting $(\pi(g)f)(x) = f(g^{-1} \cdot x)$ defines a linear representation of G on $L^2(X)$ (counting measure).
   (b) Show that $\chi_\pi(g) = \#\mathrm{Fix}(g)$.

7. In the context of problem 6 let $G = S_n$ act on $[n] = \{1, \dots, n\}$.
   (a) Show that $\pi \simeq \mathbb{1} \oplus V$ where $\mathbb{1}$ is the trivial representation on the constant vectors and V is the orthogonal complement.
   (b) Compute the character χ of the representation on V, verify that $\langle \chi, \chi \rangle_{L^2(S_n)} = 1$ and conclude that χ is irreducible.
   (c) Use $L^2(G) \simeq \bigoplus_{\pi \in \hat{G}} \pi \boxtimes \check{\pi}$ to show that $\widehat{S_3} = \{1, \mathrm{sgn}, V\}$ (hint: dimension count).
   (d) Decompose the representation arising from the action of S_n on the set $[n]^2$ into irreducibles, and connect it to problem 5.

Example: profinite groups

8. A *partially ordered set* is a pair $(P, \leq)$ where P is a set and $\leq$ is a transitive reflexive relation (but pairs of elements need not be comparable). A *directed set* is a partially ordered set in which for any $\alpha, \beta \in P$ there if $\gamma \in P$ with $\alpha, \beta \leq \gamma$.
   (a) Show that the natural numbers with the usual order form a directed set.
   (b) Let X be a topological space and let $x \in X$. Show that the set of neighbourhoods of X ordered by reverse inclusion ($U \leq V$ if $V \subset U$) is directed.
   (c) Let G be a group. Show that the set of finite index subgroups of G ordered by reverse inclusion is directed.
   (d) We can view a directed set as a category where for every α, β $\mathrm{Mor}(\alpha, \beta)$ has a unique element if $\alpha \leq \beta$ and is empty otherwise. Show that this is indeed a category (every element has an identity morphism and composition of morphisms is transitive).

9. Fix a directed set $(P, \leq)$. An *inverse system* is an assignment for each $\alpha \in P$ of a mathematical object F_α, and for each $\alpha \leq \beta$ a morphism $f_{\alpha\beta} \in F_\beta \to F_\alpha$ so that $f_{\alpha\alpha}$ is the identity and that if $\alpha \leq \beta \leq \gamma$ we have $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$.
   DEF The *inverse limit* of an inverse system of groups is the group

$$\varprojlim_{\alpha} F_\alpha = \left\{ (g_\alpha)_\alpha \in \prod_{\alpha \in P} F_\alpha \mid \forall \alpha \leq \beta : f_{\alpha\beta}(g_\beta) = g_\alpha \right\}.$$

   SUPP Treating $(P, \leq)$ as a category, show that an inverse system in a category $\mathcal{C}$ is a contravariant morphism $F \colon P \to \mathcal{C}$; define the inverse limit using a universal property, and check that this specializes to the notion above.
   (a) Check that $F = \varprojlim_{\alpha} F_\alpha$ is a group, and that the coordinate projections $\pi_\alpha \colon F \to F_\alpha$ are compatible with the inverse system in the sense that $f_{\alpha\beta} \circ \pi_\beta = \pi_\alpha$.
   (b) Suppose the F_α are topological groups and the $f_{\alpha\beta}$ are continuous. Show that if we equip $\prod_{\alpha \in P} F_\alpha$ with the product topology then $\varprojlim_{\alpha} F_\alpha$ is a closed subgroup, hence a topological group itself.
   COR The inverse limit of a system of compact groups is compact.
   DEF Call a group *pro-$\mathcal{C}$* if it is the inverse limit of groups from class $\mathcal{C}$. Examples include *profinite* groups (inverse limits of finite groups), pro-p groups (inverse limits of finite p-groups), prosolvable groups, pronilpotent groups, proalgebraic groups, etc.
   (c)

Tensor products of locally convex vector spaces

Let X, Y be Banach spaces and let $X \otimes Y$ be their *algebraic* tensor product.

11. A *cross norm* on $X \otimes Y$ is a norm such that

$$\begin{aligned} \forall x \in X, y \in Y \quad &: \quad \|x \otimes y\| \leq \|x\|_X \|y\|_Y \\ \forall x' \in X', y' \in Y' \quad &: \quad \|x' \otimes y'\| \leq \|x'\|_{X'} \|y'\|_{Y'} . \end{aligned}$$

(a) Let $\|\cdot\|$ be a cross norm. Show that equality holds above.

(a) Show that $\|t\|_\pi = \inf\{\sum_{i=1}^r \|x_i\|_X \|y_i\|_Y \mid t = \sum_{i=1}^r x_i \otimes y_i\}$ defines a cross norm on $X \otimes Y$, and that $\|t\|_\pi \geq \|t\|$ for all cross norms $\|\cdot\|$.

(b) Show that $\|t\|_\varepsilon = \sup\{|(x' \otimes y')(t)| \, x' \in X', y' \in Y, \|x'\|_{X'} = \|y'\|_{Y'} = 1\}$ defines a norm on $X \otimes Y$, and that $\|t\|_\varepsilon \leq \|t\|$ for all cross norms $\|\cdot\|$.

(c) Let $X \otimes_\varepsilon Y, X \otimes_\pi Y$ be the completions of $X \otimes Y$ with respect to these norms. Obtain a continuous inclusion $X \otimes_\varepsilon Y \hookrightarrow X \otimes_\pi Y$.

RMK In general this is not an isometry. but Grothendieck's famous inequality showed that for Hilbert spaces this is an isomorphism.