Lior Silberman's Math 535, Problem Set 1a: Topology

Topological groups: review of group theory and point set topology

1. Let G be a topological group.
   (a) Let $H \subset G$ be a subgroup. Show that its topological closure $\bar{H}$ is a subgroup as well.
   (b) Let H be an open subgroup. Show that each coset gH is open, that H is closed, and that the quotient topology on G/H is discrete.
   (c) Let $G^\circ \subset G$ be the connected component of the identity. Show that $\left\{xy^{-1} \mid x,y \in G^\circ\right\}$ is connected and conclude that G° is a subgroup of G.
   (d) Show that G° is a closed normal subgroup of G, and that its cosets are exactly the connected components of G.

   DEF The group $\pi_0(G) = G/G^\circ$ is called the *group of components* of G.

2. (Basics) Let G, H be topological groups and let $N < G$ be a subgroup.
   (a) Show that the quotient space G/N (always equipped with the quotient topology) is Hausdorff iff N is closed.
   (b) Suppose N is closed and normal in G. Show that G/N is a topological group.
   (c) Let $f \in \mathrm{Hom}(G,H)$ be a homomorphism of topological groups with kernel N, and let $q\colon G \to G/N$ be the quotient map. Show that the unique homomorphism $\bar{f}\colon G/K \to H$ such that $f = \bar{f} \circ q$ is continuous.

   RMK $\bar{f}$ need not be an isomorphism of G/K and $\mathrm{Im}(f)$ as topological groups – the topology of G/K may be finer.

3. (Direct products)
   (a) Let $\{G_i\}_{i\in I}$ be topological groups. Equip the Cartesian product $\prod_{i\in I} G_i$ with the Tychonoff topology and the direct product group structure. Show that $\prod_{i\in I} G_i$ is a topological group.
   (b) Show that $\prod_{i\in I} G_i$ is a product object in the category of topological groups.

Covering groups

4. Let G be a topological space with marked point $e \in G$ and suppose we have a continuous map $\mu\colon G \times G \to G$ such that $\mu(g,e) = \mu(e,g) = g$ for all $g \in G$. Show that $\pi_1(G,e)$ is a abelian.

5. Let G be a connected, locally path connected topological group.
   (a) (Concrete covering group) In the realization of the the universal cover $\tilde{G}$ as the space of continuous maps $\gamma\colon [0,1] \to G$ with $\gamma(0) = e$ modulu homotopy with fixed endpoints, show that setting $(\gamma_1 \cdot \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$ endows $\tilde{G}$ with the structure of a topological group with the covering map being a group homomorphism.
   (b) (continuation) Suppose $\gamma\colon [0,1] \to G$ is closed, in that $\gamma(0) = \gamma(1) = e$. Show that the image of γ in $\tilde{G}$ is central, giving a related proof that $\pi_1(G,e)$ is abelian, and that this image of $\pi_1(G,e)$ in $Z(\tilde{G})$ is discrete.
   (c) (Abstract covers) Let $p\colon H \to G$ be a covering map with H connected, and fix $e_H \in p^{-1}(e_G)$. Let $f\colon G \times G \to G$ be $f(x,y) = xy^{-1}$. Define $\bar{f}\colon H \times H \to G$ by $\bar{f} = f \circ (p \times p)$ and show that the image of $\bar{f}_*\colon \pi_1(H \times H, e_H \times e_H) \to \pi_1(G,e)$ is exactly $p_*(\pi_1(H,e_H))$. Conclude that $\bar{f}$ lifts to a continuous map $f_H\colon H \times H \to H$ and use f_H to endow H with the structure of a topological group for which p is a homomorphism. Conclude that $p^{-1}(e_G)$

is a subgroup of H, and use the action of $\pi_1(G,e)$ on $p^{-1}(e_G)$ by deck transformations to show that this subgroup is central.

RMK We will show later that $\pi_1(\mathrm{SL}_2(\mathbb{R})) \simeq \mathbb{Z}$ so this group has non-trivial covers. On the other hand (though we won't show this) none of these covers have a non-trivial finite-dimensional representation. It will follow that that covering groups are *non-algebraic* – they cannot be realized as closed subgroups of $\mathrm{GL}_n(\mathbb{C})$ for any n.

Locally compact groups

DEFINITION. Say the topological space X is *locally compact* if every point of x has a relatively compact neighbourhood.

DEFINITION. Let X,Y be topological spaces. The *compact-open topology* on $C(X,Y)$ is the topology generated by the open sets defined for $K \subset X$ compact and $U \subset Y$ open by $V(K,U) = \{f \in C(X,Y) \mid f(K) \subset U\}$ (that is, $V \subset C(X,Y)$ is open iff for all $f \in V$ there are compact $\{K_i\}_{i=1}^n$ and open $\{U_i\}_{i=1}^n$ such that $f \in \bigcap_{i=1}^n V(K_i,U_i) \subset V$).

SUPP (=supplementary, not for submission) Show that this defines a topology on $C(X,Y)$. Show that the evaluation map $C(X,Y) \times X \to Y$ is continuous in this topology if X is locally compact.

6. Let the abstract group G act on the locally compact space X by continuous maps, and suppose that the action is effective: every non-identity element of G moves some element of X. Show that G equipped with the compact-open topology is a topological group.

7. Let A be a locally compact abelian group and let $\hat{A} = \mathrm{Hom}\left(A, S^1\right)$ (by default, maps of topological groups are continuous).
   (a) Show that $\hat{A}$ is an abelian group under the operation $(\sigma+\chi)(a) \stackrel{\text{def}}{=} \sigma(a)\chi(a)$.
   (b) Show that $\hat{A}$ is locally compact when it is equipped with the compact-open topology.
   (c) Show that $\hat{A}$ is discrete iff A is compact.
   (d) *Evaluation* gives an injection $A \hookrightarrow \hat{\hat{A}}$. Show that this map is continuous.
   RMK It is a fact (Pontryagin duality) that this is a topological isomorphism.