Math 535: Real Groups
Lecture Notes

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These are rough notes for the Spring 2023 course, version as of March 31, 2023.

Contents

Introduction

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0.1. Administrivia

- Problem sets will be posted on the course website.
  - To the extent I have time, solutions may be posted on the LMS.
- Textbooks
  - Warner, Lee
  - Bröcker–tom Dieck, *Representations of Compact Lie Groups* (GTM-98)
  - Knapp, *Lie groups beyond an introduction*
  - Knapp, *Representation Theory of Semisimple Groups*
- No exams.

0.2. Background

CHAPTER 1

Basics: Locally compact groups and their representations

Motivation

A Lie group is a smooth manifold equipped with a compatible group structure. In particular it is a *topological space* equipped with a compatible group structure. Before investigating Lie groups and their (continuous!) representations we will say a little on general topological groups and their representations. The main result will be the Peter–Weyl Theorem, codified in Theorems 36, 38, describing the representation theory of a compact group in terms of its unitary dual.

1.1. Topological groups

DEFINITION 1. A *topological group* is a group object in the category of Hausdorff topological spaces. A *homomorphism* of topological groups is a continuous group homomorphism. An *action* of the topological group G on the topological space X is a group action $\cdot\colon G \times X \to X$ which is continuous for the product topology on $G \times X$.

Note that the regular action of G on itself is a continuous action by homeomorphisms.

EXAMPLE 2. $\mathbb{R}$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{Q})$, $\mathbb{Q}_p$, C_2^X (X arbitrary!), etc..

LEMMA 3. *Suffices to assume T_1, that is that $\{e\} \subset G$ is closed.*

PROOF. By the invariance of the topology if $\{e\}$ is closed so is every point, and it is enough to separate e from g for every $g \neq e$. Since the group is T_1, the set $G \setminus \{g\}$ is open. By continuity of the map $(x,y) \mapsto xy^{-1}$ at the identity there is a neighbourhood $(e,e) \in U \times V \subset G \times G$ such that $xy^{-1} \neq g$ for al $(x,y) \in U \times V$. Then $W = U \cap V$ works. □

LEMMA 4. *Let $H \subset G$ be a subgroup. Then the quotient topology on G/H is Hausdorff iff H is closed.*

PROOF. Exercise. □

REMARK 5. *Algebraic groups* are (generally) not topological groups. This directly follows from the fact that the Zariski topology is not Hausdorff, but more fundamentally is related to the fact that Zariski topology on the product of two k-varieties $X \times_k Y$ is *not* the product topology.

1.2. Representation Theory

1.2.1. Continuous representations.

DEFINITION 6. A *representation* π of the topological group G on the TVS V_π is a continuous action by linear maps. A *unitary* representation is a represetation on a Hilbert space V_π by unitary maps.

DEFINITION 7. Let (π, V) and (σ, W) be representations of G. An *intertwining operator* (or *G-homomorphism*) between them is a continuous map $f\colon V \to W$ such that

$$\forall g \in G: \sigma(g) \circ f = f \circ \pi(g).$$

We will write $\mathrm{Hom}_G(V,W)$ for the set of G-homomorphisms, $\mathrm{Rep}(G)$ for the category of representations and G-homomorphisms.

LEMMA 8. *Let $(\pi, V) \in \mathrm{Rep}(G)$. If $W \subset V$ is G-invariant then so is its closure $\bar{W}$.*

DEFINITION 9. Call (π, V) (topologically) *irreducible* if its only closed G-invariant subspaces are the obvious ones.

EXAMPLE 10. Fix a group G. Representations of G include:

(1) The *trivial representation* is the unique representation with $V = \{\underline{0}\}$.
(2) The regular representation on any function space on G, including $C(G)$, $L^p(G)$ (if G is locally compact) etc. If $p = 2$ we obtain a unitary representation. If G is a manifold (see later) we similar have actions on $C^\infty(G), C^\omega(G)$ and also on appropriate Sobolev spaces.
(3) Simlarly for function spaces on G/H, and even more generally on topological G-spaces X.

EXAMPLE 11. Let $O(n)$ act on S^{n-1}, hence on various function spaces. It also preserves the space of polynomial functions $\mathbb{R}[x_1, \ldots, x_n]$ (restrictions of polynomials to the sphere) which is the direct sum of invariant finite-dimensional subspaces: $\mathbb{R}[\underline{x}]^{=d}$ (homogenous of degree d). This direct sum is dense.

1.2.2. Constructions.

LEMMA-DEFINITION 12. *Let (π, V) and (σ, W) be representations of G. Then the following are continuous representations:*

(1) *$(\check{\pi}, V')$ where V' is the topological dual (say with the weak-* topology) $\check{\pi}(g) = \left(\pi(g)^{-1}\right)'$ (dual map of $\pi(g)^{-1} = \pi(g^{-1})$).*
(2) *$(\pi \oplus \sigma, V \oplus W)$ where we equip $V \oplus W$ with the product topology and set $(\pi \oplus \sigma)(g) = \pi(g) \oplus \sigma(g)$.*
(3) *$(\bar{\pi}, V/U)$ where $U \subset V$ is a G-invariant closed subspace and $\bar{\pi}(g)(\underline{v} + U) = \pi(g)\underline{v} + U$.*

PROOF. Exercise. $\square$

LEMMA-DEFINITION 13 (Naive tensor product). *Let $(\pi, V), (\sigma, W)$ be representations of G, H respectively. Then $G \times H$ acts on the algebraic tensor product $V \otimes W$ by $(\pi \otimes \sigma)(g,h) \stackrel{def}{=} \pi(g) \otimes \sigma(h)$.*

REMARK 14. When V, W are finite-dimensional so is $V \otimes W$ and there is no problem with the topology. In the infinite-dimensional case the situation is much more complicated (c.f. Grothendieck).

1.2.3. Matrix coefficients.

DEFINITION 15. Let (π, V) be a representation of G. A *matrix coefficient* of V is any function

$$\Phi_{\underline{v},\underline{v}'}(g) = \left\langle \pi(g)\underline{v}, \underline{v}' \right\rangle$$

where $\underline{v} \in V$, $\underline{v}' \in V'$.

REMARK 16. It is always the case that $\Phi_{\underline{v},\underline{v}'} \in C(G)$. Further analytic properties of the matrix coefficients (smoothness and decay) are very important.

LEMMA 17. *The map* $(\underline{v},\underline{v}') \mapsto \Phi_{\underline{v},\underline{v}'}$ *is bilinear; the resulting map* $V \otimes \check{V} \to C(G)$ *is an (algebraic) intertwining operator where* $G \times G$ *acts on* $C(G)$ *by* $((g_1,g_2)\cdot f)(x) = f\left(g_2^{-1} x g_1\right)$.

PROOF. We only prove the last claim:

$$\begin{aligned}
\Phi_{\pi(g_1)\underline{v},\check{\pi}(g_2)\underline{v}'}(x) &= \left\langle \pi(x)\pi(g_1)\underline{v}, {}^t\pi(g_2^{-1})\underline{v}' \right\rangle \\
&= \left\langle \pi(g_2^{-1})\pi(x)\pi(g_1)\underline{v}, \underline{v}' \right\rangle \\
&= \left\langle \pi(g_2^{-1} x g_1)\underline{v}, \underline{v}' \right\rangle \\
&= \Phi_{\underline{v},\underline{v}'}(g_2^{-1} x g_1).
\end{aligned}$$

□

REMARK 18. We see that abstract representations have *concerete models*.

DEFINITION 19. Say that an irrep (π,V) of a locally compact group belongs to the *discrete series* if it is isomorphic to an irreducible subrepresentation of the regular representation on $L^2(G)$.

EXAMPLE 20. Suppose (π,V) is *unitarizable*, in that there is a G-invariant continuous Hermitian product on V (so that the completion is a Hilbert space). Equipping V' with the dual inner product, which is also invariant, we see that the matrix coefficients of π are bounded.

1.3. Compact groups: the Peter–Weyl Theorem

In this section G is a compact group, equipped with its probability Haar measure dg.

1.3.1. Finite-dimensional representations: Schur orthogonality. Fix a representation (π,V) of G where V is finite-dimensional.

LEMMA 21 (Unitarity). *There is a* G*-invariant Hermitian product on* V.

PROOF. Let $(\cdot,\cdot)$ be any Hermitian product on V, and for $\underline{u},\underline{v} \in V$ set

$$\langle \underline{u},\underline{v} \rangle = \int_G (\pi(g)\underline{u}, \pi(g)\underline{v})\,\mathrm{d}g$$

where dg is the probability Haar measure on G. □

COROLLARY 22. *Let* $W \subset V$ *be an invariant subspace. Then it has a* complement*: another invariant subspace* $W^\perp$ *such that* $V = W \oplus W^\perp$.

PROOF. Take the orthogonal complement wrt an invariant Hermitian product. □

The following should be compared with the spectral theorem.

THEOREM 23 (Maschke). *Every finite-dimensional representation is a direct sum of irreducible subspaces.*

PROOF. Let $U \subset V$ be maximal wrt inclusion among all subspaces which are direct sums of irreducibles. If $U \neq V$ then $U^\perp$ is non-trivial; let $W \subset U^\perp$ be a non-zero invariant subspace of minimal dimension. Then W is necessarily irreducible and $U \oplus W$ is the direct sum of irreducibles, a contradiction. □

PROBLEM 24. How does the unitary structure interact with our abstract representaiton theory from before? In particular, does our notion of isomorphism change?

PROPOSITION 25 (Schur's Lemma). *Let* $(\pi,V),(\sigma,W)$ *be finite-dimensional irreducible representations of G. Then* $\mathrm{Hom}_G(V,W) \simeq \begin{cases} \mathbb{C} & \pi \simeq \sigma \\ 0 & \pi \not\simeq \sigma \end{cases}$.

PROOF. Since the kernel and image of an intertwining operator are invariant subspaces, any non-zero G-homomorphism from an irrep is injective and to an irrep is surjective. In particular, if π,σ are non-isomorphic they support no non-zero maps between them. It remains to compute $\mathrm{Hom}_G(V,V)$. For this let $T \in \mathrm{Hom}_G(V,V)$, so that $\pi(g)T = T\pi(g)$ for all $g \in G$. Since $\mathbb{C}$ is algebraically closed, T has at least one eigenvalue λ; let $V_\lambda = \mathrm{Ker}(T-\lambda)$, a non-trivial subspace of V. Then for any $\underline{v} \in V_\lambda$ we have $(T-\lambda)(\pi(g)\underline{v}) = \pi(g)((T-\lambda)\underline{v}) = \underline{0}$ so that $\pi(g)\underline{v} \in V_\lambda$ as well. It follows that $V_\lambda \subset V$ is a G-invariant subspace, and hence that $V_\lambda = V$ and $T = \lambda\,\mathrm{Id}$. □

OBSERVATION 26. *Every matrix cofficient ofa continuous representation is a continuous function on the compact space G, hence square-integrable.*

PROPOSITION 27 (Schur Orthogonality). *Let* $(\pi,V),(\sigma,W) \in \mathrm{Rep}(G)$ *be finite-dimensional irreps.*

(1) *If* π,σ *are non-isomorphisc then any two matrix coefficients of* π,σ *are orthogonal.*
(2) *Let* $d_\pi = \dim V_\pi$. *Then for any* $\underline{v},\underline{w} \in V$ *and* $\underline{v}',\underline{w}' \in V'$ *we have*

$$\left\langle \Phi^\pi_{\underline{u},\underline{u}'}, \Phi^\pi_{\underline{v},\underline{v}'} \right\rangle_{L^2(G)} = \frac{1}{d_\pi} \left\langle \underline{v},\underline{u}' \right\rangle \left\langle \underline{u},\underline{v}' \right\rangle$$

PROOF. Let $T\colon V \to W$ be any linear map, and let

$$\bar{T} = \int_G \sigma(g^{-1})T\pi(g)\,\mathrm{d}g\,.$$

Then

$$\begin{aligned}\bar{T}\pi(h) &= \int_G \sigma(g^{-1})T\pi(gh)\,\mathrm{d}g \\ &= \int_G \sigma(hg^{-1})T\pi(g)\,\mathrm{d}g \\ &= \sigma(h)\bar{T}\,.\end{aligned}$$

It follows that $\bar{T} \in \mathrm{Hom}_G(V,W)$. Next, for any $\underline{v} \in V$, $\underline{v}' \in V'$, $\underline{w} \in W$, $\underline{w}' \in W'$ let $T = |\underline{w}\rangle\langle\underline{v}'|$. Then

$$\begin{aligned}\left\langle \underline{w}' \right| \bar{T} \left| \underline{v} \right\rangle &= \int \left\langle \underline{w}' \right| \sigma(g^{-1}) \left| \underline{w} \right\rangle \left\langle \underline{v}' \right| \pi(g) \left| \underline{v} \right\rangle \mathrm{d}g \\ &= \int_G \mathrm{d}g\, \overline{\left\langle \underline{w} \right| \sigma(g) \left| \underline{w}' \right\rangle} \left\langle \underline{v}' \right| \pi(g) \left| \underline{v} \right\rangle \\ &= \left\langle \Phi^\sigma_{\underline{w}',\underline{w}}, \Phi^\pi_{\underline{v},\underline{v}'} \right\rangle_{L^2(G)}\,,\end{aligned}$$

where we have identified W' with W via the Riesz representation theorem and the inner product.

(1) Suppose π,σ are non-isomorphic. Then $\bar{T} = 0$ and the two matrix coefficients are orthogonal.

(2) Suppose $V = W$, $\pi = \sigma$. Then $\bar{T} = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Normalizing the Haar measure on G to be a probability measure, we see that $\bar{T}$ is the average of conjugates of T so

$$d_\pi \lambda = \mathrm{Tr}\,\bar{T} = \mathrm{Tr}\,T = \left\langle \underline{v}', \underline{w} \right\rangle .$$

Solving for λ it follows that

$$\begin{aligned}\left\langle \Phi^\pi_{\underline{w}',\underline{w}}, \Phi^\pi_{\underline{v},\underline{v}'} \right\rangle_{L^2(G)} &= \left\langle \underline{w}' \middle| \bar{T} \middle| \underline{v} \right\rangle = \lambda \left\langle \underline{w}' \middle| \mathrm{Id} \middle| \underline{v} \right\rangle \\ &= \frac{1}{d_\pi} \left\langle \underline{w}', \underline{v} \right\rangle \left\langle \underline{v}', \underline{v} \right\rangle .\end{aligned}$$

□

COROLLARY 28. $\langle \chi_\pi, \chi_\sigma \rangle_{L^2(G)} = \delta_{\pi \simeq \sigma}$.

COROLLARY 29. *For each finite-dimensional irrep π let $\mathcal{C}(\pi)$ be the space of matrix coefficients of π. Then*

$$\bigoplus_\pi \mathcal{C}(\pi) \subset L^2(G)$$

is an orthogonal direct sum.

1.3.2. Infinite-dimensional representations and the Peter–Weyl Theorem. Let (π, V) be a continuous representation of the locally compact group G on the quasi-complete locally convex TVS V.

LEMMA-DEFINITION 30. *TFAE for $\underline{v} \in V$, in which case we call it G-finite*

(1) $\mathrm{Span}_{\mathbb{C}} \left\{\pi(g)\underline{v}\right\}_{g \in G} \subset V$ *is finite-dimensional.*
(2) *There is a finite-dimensional G-invariant subspace $W \subset V$ with $\underline{v} \in W$.*

Furthermore, the set V_G of G-finite vectors is a G-invariant algebraic subspace of V.

PROOF. Given (1), set $W = \mathrm{Span}_{\mathbb{C}} \left\{\pi(g)\underline{v}\right\}_{g \in G}$ to get (2). Given (2), $\mathrm{Span}_{\mathbb{C}} \left\{\pi(g)\underline{v}\right\}_{g \in G} \subset W$ for all G-invariant subspaces W containing $\underline{v}$. Finally, if $\underline{v}_1, \underline{v}_2 \in V_G$, say with $\underline{v}_i \subset W_i$ with W_i G-inv't and f.d. then $\alpha\underline{v}_1 + \underline{v}_2 \in W_1 + W_2$ which is G-inv't and f.d. □

PROPOSITION 31. *In a compact group G we have $\bigoplus_\pi \mathcal{C}(\pi) = L^2(G)_K$, where G acts on $L^2(G)$ via the right-regular representation $(R_g f)(x) = f(xg)$.*

PROOF. Since each $\mathcal{C}(\pi)$ is finite-dimensional, their algebraic direct sum is contained in $C(G)_K \subset L^2(G)_K$. Conversely, let $W \subset L^2(G)$ be a right-G-invariant finite-dimensional subpsace. By Maschke's Theorem 23, W is the direct sum of irreducible subpsaces so without loss of generality it suffices to show $W \subset \bigoplus_\pi \mathcal{C}(\pi)$ for irreducible W.

Thus let $\{f_i\}_{i=1}^d \subset W$ be an orthonormal basis. Then for $f \in W$ and $g \in G$ we have $R_g f \in W$ and hence

$$R_g f = \sum_{i=1}^d a_i(g) f_i$$

for some $a_i(g) \in \mathbb{C}$. In fact,

$$a_i(g) = \left\langle f_i, R_g f \right\rangle_{L^2(G)} = \Phi^W_{f_i,f}(g) \in \mathcal{C}(W)$$

and we conclude that for *fixed* g

$$R_g f = \sum_{i=1}^{d} \Phi^W_{f_i,f}(g) f_i$$

(the sum in $W \subset L^2(G)$). In other words, given g it holds for almost every $x \in G$ that

$$f(xg) = \sum_{i=1}^{d} \Phi^W_{f_i,f}(g) f_i(x)\,.$$

If the identity held for *all* x we could set $x = e$ and write f as a linear combination of matrix coefficients. To get around this difficulty consider both sides as functions on $G \times G$. Now both sides are in $L^2(G \times G)$, so by Fubini they are equal a.e. Applying Fubini in the other order it follows that for almost every $x \in G$ we have $f(xg) = \sum_{i=1}^{d} \Phi^W_{f_i,f}(g) f_i(x)$ for almost every $g \in G$, and that is the desired claim. $\square$

DEFINITION 32 (Topological group ring). For $f \in C_c(G)$ and $\underline{v} \in V$ set $\pi(f)\underline{v}$ by

$$\pi(f)\underline{v} = \int_G f(g)\pi(g)\underline{v}\,\mathrm{d}g\,.$$

LEMMA 33. *$\pi(f)\colon V \to V$ is a continuous linear map, and $f \mapsto \pi(f)$ is a continuous algebra homomorphism $C_c(G) \to \mathrm{End}(V)$ where $C_c(G)$ is equipped with the convolution product and the direct limit topology.*

PROOF. Scaling, we may assume $|f(g)| \le 1$ for all g. Let $U \subset V$ be a closed convex neighbourhood of zero. Then for each $g \in \mathrm{supp}(f)$ there are neighbourhoods $g \in W_g \subset G$ and (convex) $\underline{0} \in U_g \subset V$ such that $\pi(x)\underline{u} \in \frac{1}{\mathrm{vol}\,\mathrm{supp}(f)} U$ for all $x \in W_g$, $\underline{u} \in U_g$. Covering $\mathrm{supp}(f)$ with $\cup_{i=1}^{r} W_{g_i}$ and setting $\bar{U} = \cap_{i=1}^{r} U_{g_i}$ we see that for all $g \in \mathrm{supp}(g)$ and $\underline{v} \in \bar{U}$, $f(g)\pi(g)\underline{v} \in \frac{1}{\mathrm{vol}\,\mathrm{supp}(f)} U$. It follows that $\pi(f)\underline{v} \in U$.

Rest proved similarly. $\square$

COROLLARY 34. *Let $\{f_n\} \subset C_c(G)$ be an approximate identity. Then $\pi(f_n)\underline{v} \to \underline{v}$.*

EXAMPLE 35 (Smoothing). Let $V \subset L^2(G)$ be a closed G-invariant subspace. Then $V \cap C(G)$ is dense in G.

PROOF. It suffices to show that $\pi(f)\varphi \in C(G)$ for each $f \in C_c(G)$, $\varphi \in L^2(G)$. Indeed,

$$\begin{aligned}(\pi(f)\varphi)(x) &= \int f(g)\varphi(g^{-1}x)\,\mathrm{d}g \\ &= \int f(xg)\varphi(g^{-1})\,\mathrm{d}g\end{aligned}$$

so that

$$\begin{aligned}|(\pi(f)\varphi)(x) - (\pi(f)\varphi)(y)| &= \left|\int \delta(g)\left(f(xg^{-1}) - f(yg^{-1})\right)\varphi(g)\,\mathrm{d}g\right| \\ &\le \left\|\delta(g)\left(f(xg^{-1}) - f(yg^{-1})\right)\right\|_{L^2(G)} \|\varphi\|_{L^2(G)} \\ &\xrightarrow[y\to x]{} 0\end{aligned}$$

since f is uniformly continuous and δ is bounded on any compact set.

Suppose now that G is compact. $\square$

THEOREM 36 (Peter–Weyl I). *We have*

$$L^2(G) = \hat{\bigoplus_{\pi}} \mathcal{C}(\pi)\,.$$

PROOF. Let $V = \left(\bigoplus_\pi \mathcal{C}(\pi)\right)^\perp$ and note that V is a subrepresentation of $\left(L^2(G), L\times R\right)$. If $V \neq \{0\}$ let $f \in V$ be non-zero, and by continuity of the left G-action on $L^2(G)$ let $U \subset G$ be a symmetric, conjugation-invariant neighbourhood of 1 such that $\|L_u f - f\|_2 \leq \frac{1}{2}\|f\|$. Let $\chi \in C_c(U)$ be positive, satisfy $\chi(u) = \chi(u^{-1})$, integrate to 1 and be conjugation invariant. Then $\|L(\chi)f - f\|_2 \leq \frac{1}{2}\|f\|$ and in particular $L(\chi)\colon V \to V$ is a non-zero operator. It is also self-adjoint and compact. By the spectral theorem its eigenspaces are finite-dimensional; they are also R_g-invariant and it follows that V contains G-finite vectors, a contradiction. $\square$

COROLLARY 37 (Peter–Weyl II). $\bigoplus_\pi C_c(\pi)$ *is dense in* $C(G)$.

PROOF. Since the product of matrix coeffs is a matrix coefficient of the tensor product, this is a subalgebra closed under complex conjugation and it suffices to show it separates the points. By G-invariance it suffices to separates points from the identity.

For this consider $\bigcap_\pi \mathrm{Ker}(\pi)$. Every $f \in L^2(G)$ is invariant by this closed subgroup, so it's trivial. It follows that for any $g \in G$ there is π such that $\pi(g) \neq \mathrm{id}$. Let $\underline{v} \in V_\pi$ be of norm 1 such that $\pi(g)\underline{v} \neq \underline{v}$. Then by unitarity $\langle \pi(g)\underline{v}, \underline{v}\rangle \neq 1$ and hence

$$\Phi_{\underline{v},\underline{v}}(g) \neq 1 = \Phi_{\underline{v},\underline{v}}(e)\,.$$

$\square$

THEOREM 38 (Peter–Weyl II). *Every irrep of G is finite-dimensional; for any representation* V_G *is dense in* V.

PROOF. Clearly the second assertion implies the first. We first note that the argument of Theorem 36 shows that $\{\pi(\chi)\underline{v} \mid \underline{v} \in V, \chi \in C(G)\}$ is dense in V, and its corollary shows that $C(G)_G$ is dense in $C(G)$. By the continuity of Lemma 33 this means that $\{\pi(\chi)\underline{v} \mid \underline{v} \in V, \chi \in C(G)_G\}$ is dense in V, and this space consists of G-finite vectors. $\square$

CHAPTER 2

Lie Groups and Lie Algebras

Motivation

A Lie group is a smooth manifold equipped with a compatible group structure. In particular it is a *manifold*. We therefore being by defininig what we mean by manifolds,

2.1. Smooth manifolds

2.1.1. Manifolds.

DEFINITION 39. Let $U \subset \mathbb{R}^n$ be open. Then $C^\infty(U;\mathbb{R}^m)$ is the set of infinitely differentiable $\mathbb{R}^m$-valued functions on U.

DEFINITION 40. A *manifold* is a space $(M,\mathcal{F})$ where M is a topological space and $\mathcal{F}$ is a sheaf on M which is locally isomorphic to the sheaf of smooth functions on an open subset of $\mathbb{R}^n$.

DEFINITION 41. A *coordinate chart* (or *patch)* in a topological space M is a pair (U,φ) where $U \subset M$ is open and $\varphi \colon U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. Two coordinate patches $(U_1,\varphi_1),(U_2,\varphi_2)$ are *compatible* if $\varphi_1 \upharpoonright_{U_1\cap U_2} \circ (\varphi_2 \upharpoonright_{U_1\cap U_2})^{-1}$ is a smooth map.
An *altas* on M is a covering of M by compatible coordinate patches. A *smooth manifold* is a pair $(M,\mathcal{A})$ where M is a second countable topological space and $\mathcal{A}$ is an atlas on M.

EXAMPLE 42. $\mathbb{R}^n$, S^n, $\mathbb{T}^n$.

LEMMA 43. *If two charts are compatible with an atlas they are compatible with each other.*

COROLLARY 44. *Every atlas is contained in a maximal atlas, namely the set of all charts compatible with the given atlas.*

DEFINITION 45. A maximal atlas is also known as a *smooth structure* on M.

EXAMPLE 46. Exotic spheres.

LEMMA 47. *If $m \neq n$ $\mathbb{R}^m,\mathbb{R}^n$ are not locally homeomorphic so for a connected manifold the dimension need not be assumed constant.*

DEFINITION 48. Let M,N be smooth manifolds. A map $f \colon M^m \to N^n$ is *smooth* if for every charts (U,φ) of M and (V,ψ) of N, $\psi \circ f \circ \varphi^{-1}$is smooth.

LEMMA 49. *Composition of smooth maps is smooth.*

2.1.2. Tangent and contagent spaces. Fix a vector space k.

DEFINITION 50. A *Lie algebra* over k is a k-vector space $\mathfrak{g}$ equipped with a bilinear form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:
(1) (alternating) $[X,X] = 0$

(2) (Jacobi identity) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$.

EXAMPLE 51 (Standard constructions). Let A be an associative k-algebra. We get two natural Lie algebras from it:

(1) A itself, equipped with $[a,b]=ab-ba$.
(2) Call $d\in\mathrm{End}_{k\text{-vsp}}(A)$ a *derivation* if $d(ab)=d(a)b+ad(b)$. Then the space $\mathcal{D}_A$ of derivations is a Lie subalgebra of $\mathrm{End}_{k\text{-vsp}}(A)$.
(3) One canonical example: $A=C^\infty(M)$; then $\mathcal{D}_M\stackrel{\text{def}}{=}\mathcal{D}_{C^\infty(M)}$ is called the set of (smooth) *vector fields* on M.

LEMMA 52 (Localization of vector fields). *Let $X\in\mathcal{D}_M$, $f,g\in C^\infty(M)$.*

(1) *Let f be constant. Then $Xf\equiv 0$.*
(2) *Let $f(p)=g(p)=0$. Then $(X(fg))(p)=0$.*
(3) *Let f be constant in a neighbourhood of p. Then $(Xf)(p)=0$. In particular, if $f=g$ in a neighbourhood of p then $Xf(p)=Xg(p)$.*

PROOF. Say $f(p)=1$ for all x. Then $Xf=X\left(f^2\right)=2f\cdot Xf=2Xf$. It follows that $Xf\equiv 0$. Simliarly, if $f(p)=g(p)=0$ then $(X(fg))(p)=(Xf(p))(g(p))+(f(p))(Xg(p))=0$.

Let U be a neighbourhood of $p\in U$ and suppose $f\restriction_U\equiv 1$. Choose $g\in C^\infty_{\mathrm{c}}(U)$ such that $g(p)\neq 0$. Since $fg=g$ we have $Xf\cdot g+f\cdot Xg=Xg$, Evaluating at p we get $Xf(p)g(p)=0$ so $Xf(p)=0$. □

LEMMA-DEFINITION 53. $I_p=\{f\in C^\infty(M)\mid f(p)=0\}$ *is a maximal ideal of* $C^\infty(M)$.

LEMMA-DEFINITION 54 (Hadamard). *The* contagent space $T^*_pM=I_p/I_p^2$ *is a vector space of dimension n and $\bigcup_{p\in M}T^*_pM$ is a vector bundle, the* contagent bundle.

PROOF. Let f vanish in a neighbourhood U of p, and let $g\in C^\infty_{\mathrm{c}}(U)$ vanish at p as well. Then $f=fg\in I_p^2$. It follows that $f,g\in C^\infty(M)$ agree in a neighbourhood of p then $f-g\in I_p^2$. We can now work locally, in particular near $\underline{0}\in\mathbb{R}^n$. We next show that every class in I_p/I_p^2 has a linear representative. Indeed let f be smooth in a neighbourhood of $\underline{0}\in\mathbb{R}^n$ and set $g(t)=f(t\underline{x})$. Then

$$\begin{aligned}
f(\underline{x})-f(\underline{0})=g(t)-g(1)&=\int_0^1 g(t)\,\mathrm{d}t\\
&=\int_0^1 \underline{x}\cdot\nabla f(t\underline{x})\,\mathrm{d}t\\
&=\sum_{i=1}^n x_i\cdot\int_0^1\frac{\partial}{\partial x^i}f(t\underline{x})\,\mathrm{d}t\\
&=\nabla f(\underline{0})\cdot\underline{x}+\sum_{i=1}^n x_ih_i
\end{aligned}$$

where $h_i(\underline{x})=\int_0^1\frac{\partial}{\partial x^i}f(t\underline{x})\,\mathrm{d}t-\frac{\partial f}{\partial x^i}(\underline{0})\in I_{\underline{0}}$. It follows that

$$f(\underline{x})-f(\underline{0})-\nabla f(\underline{0})\cdot\underline{x}\in I_p^2.$$

To see that the linear functions inject into I_p/I_p^2 (so that the dimension is n) note that each linear function has a non-zero directional derivative, but that operation is a derivation in $C^\infty(U)$ ($U\subset\mathbb{R}^n$) and vanishes on elements of I_p^2. □

LEMMA-DEFINITION 55 (The tangent space). *The linear dual* $T_pM = \mathrm{Hom}_{\mathbb{R}}\left(T_p^*M, \mathbb{R}\right)$ *is called the* tangent space. The resulting bundle is called the *tangent bundle*.

(1) *The pairing* $(X, f) \mapsto Xf(p)$ *associates to each vector field* X *a linear functional on* T_p^*X.
(2) *The resulting map* $\mathcal{D}_M \to \left(T_p^*M\right)'$ *is surjective.*

CONCLUSION 56. In local coordinates, a vector field is an operator of the form $\sum_{i=1}^n a_i(\underline{x})\frac{\partial}{\partial x^i}$.

EXERCISE 57. T_pM is also the space of derivations on the algebra of germs of smooth functions at p.

PROPOSITION 58 (Canonical sheaf). (1) *Let* X *be a vector field on* M, $U \subset M$ *an open set. For* $f \in C^\infty(U)$ *and* $p \in U$ *let* $h \in C_c^\infty(U)$ *such that* $h \equiv 1$ *near* p *and set* $(X \restriction_U f)(p) = (X(hf))(p)$ *(note that* $hf \in C_c^\infty(M)$*). Then* $X \restriction_U$ *is a well-defined vector field on* U *and* $X \mapsto X \restriction_U$ *is a map of lie algebras.*
(2) *(Patching) Let* $\{U_i\}_{i\in I}$ *be an open cover of* M. *Let* X, Y *be a vector fields on* M *and suppose that* $X \restriction_{U_i} = Y \restriction_{U_i}$ *for all* i *then* $X = Y$.
(3) *(Gluing) Let* $\{U_i\}_{i\in I}$ *be an open cover of* M *and suppose given for each* i *a vector field* X_i *on* U_i *such that* $X_i \restriction_{U_i\cap U_j} = X_j \restriction_{U_i\cap U_j}$ *for all* i, j. *Then there is a vector field* X *on* M *such that* $X_i = X \restriction_{U_i}$.

2.1.3. Derivatives of maps.

LEMMA-DEFINITION 59. *Let* $\varphi\colon M \to N$ *be a smooth map. Let* $p \in M$ *and* $v \in T_pM$. *Then the map* $d\varphi_p(v)\colon C^\infty(N) \to \mathbb{R}$ *given by* $f \mapsto v(f \circ \varphi)$ *is a local derivation at* $\varphi(p)$. *It is called the* differential *of* φ. *The map* $d\varphi_p\colon T_pM \to T_{\varphi(p)}N$ *is linear and extends to a smooth map* $d\varphi\colon TM \to TN$ *compatible with* φ. *The construction is functorial (in other words, the chain rule holds).*

THEOREM 60 (Inverse and implicit function theorems). *Let* $\varphi\colon M \to N$ *be smooth.*

(1) *Suppose* $d\varphi_p$ *is injective. Then* φ *is injective in a neighbourhood of* p.
(2) *Suppose* $d\varphi_p$ *is a surjective. Then* φ *is an open map in a neighbourhood of* p.
(3) *Suppose* $d\varphi_p$ *is an isomorphism. There are open neighbourhoods of* p *and* $\varphi(p)$ *for which* φ *is a diffeomorphism.*
(4) *Suppose* $d\varphi_p$ *is surjective for* p *on a level set* $P = \varphi^{-1}(n)$. *Then the level set is a submanifold of dimension* $\dim N - \dim M$.

DEFINITION 61. A smooth map $f\colon M \to N$ is a:

(1) *Immersion* if df_p is injective for every $p \in M$.
(2) *Local embedding* if for every $p \in M$ there is a neighbourhood U such that $f \restriction_U$ is a homeomorphism onto its image (with the relative topology).
(3) An embedding if it is an injective immersion which is a homeomorphism onto its image.
(4) A *diffeomorphism* if it has a smooth inverse.

THEOREM 62 (Inverse function theorem). f *is an immersion iff it is a local embedding.*

EXAMPLE 63. The line that almost meets itself.

DEFINITION 64. A *parametrized submanifold* of N is a pair (M, f) where $f\colon M \to N$ is an injective immersion. Two parametrizations $(M_1, f_1), (M_2, f_2)$ are *equivalent* if they are conjugate by a diffeomorphism of M_1, M_2. A *submanifold* of N is an equivalence class.

If (M,f) is a parametrized submanifold N then $T(N) = f_*(TM)$ is a subbundle of $TN \restriction_M$. It is independent of the choice of parametrization. Conversely, we'd like to investigate when a choice of subspace of T_pM at each p corresponds to a submanifold.

DEFINITION 65. Let $\gamma\colon [a,b] \to M$ be a smooth curve. We then set $\dot\gamma(t) = d\gamma\left(\frac{\partial}{\partial t}\right)(t)$. We say γ is an *integral curve* of $X \in \mathcal{D}_M$ if $\dot\gamma(t) = X(\gamma(t))$ for each t.

- The Picard Theorem on ODE shows that for any X and $p \in M$ there is an integral curve of X through p living on an interval about 0, and that any two integral curves with $\gamma(0) = p$ agree on their interval of definition.

We now generalize this from 1-dimensional submanifolds to higher dimension.

DEFINITION 66. A *distribution* of dimension k on M is equivalently either of:

(1) A smooth choice of k-dimensional subspaces $V_p \subset T_pM$ for each $p \in M$.
(2) A smooth section of the Grassmanian bundle, or a subbundle of TM.
(3) For a covering set of neighbourhoods $U \subset M$ choices of vector fields $\{X_i\}_{i=1}^k \subset \mathcal{D}_U$ so that for each $p \in U$, $\{X_i(p)\} \subset T_pM$ are linearly independent and so that $V_p = \mathrm{Span}_{\mathbb{R}}\{X_i(p)\}_i$ is independent of U as long as $p \in U$.

Call a vector field $X \in \mathcal{D}_M$ a *section* of the distribution V if (1) $X_p \in V_p$ for each p iff (2) it is a section of the subbundle (3) For each U there are $a_i \in C^\infty(U)$ so that $X \restriction_U = \sum_i a_i X_i$.

DEFINITION 67. Call a submanifold $\left(N^k, \varphi\right)$ of M *tangent* to the distribution V if for each $p \in N$, $d\varphi_p$ is an isomorphism of T_pN and $V_{\varphi(p)} \subset T_{\varphi(p)}M$.

OBSERVATION 68. *Suppose $N \subset M$ is tangent to V, and let X,Y be sections of V. We can then think of X,Y as vector fields on N, so that $[X,Y]$ is a vector field on N as well. It follows that $[X,Y]$ is also a section of V.*

In fact, this necessary condition is also sufficient:

THEOREM 69 (Frobenius). *The following are equivalent for a distribution V on M:*

(1) *Through each $p \in M$ there is a unique (up to equivalence) submanifold tangnet to V; this submanifold is injectively submersed.*
(2) *The distribution is* completely integrable*: for every two sectoins X,Y of V, the vector field $[X,Y]$ is also a section.*

REMARK 70. In the local view above it suffices to check that the integrability condition on the spanning fields: $\left[X_i, X_j\right] = \sum_k a_k X_k$ for some $a_k \in C^\infty(U)$.

2.2. Lie groups

DEFINITION 71. A *Lie group* is a group object in the category of smooth manifolds, in other words a smooth manifold G together with smooth maps $\cdot\colon G \times G \to G$ and ${}^{-1}\colon G \to G$ such that $\left(G, \cdot, {}^{-1}\right)$ is an abstract group. A *homomorphism* of Lie group is an abstract homomorphism which is also a smooth map.

EXAMPLE 72. The basic example is $\mathbb{R}$, but we also have:

(1) $\mathbb{R}^n$, $(\mathbb{R}/\mathbb{Z})^n = \mathbb{R}^n/\mathbb{Z}^n$
(2) $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{R})$

(3) $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SO}(Q) = \mathrm{SO}(p,q)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$
(4) Direct and semidirect products.
(5) $\mathrm{Isom}(\mathbb{E}^n)$, $\mathrm{Isom}\,(M,g)$
(6) $\mathrm{Aff}_n(\mathbb{R})$

DEFINITION 73. An *action* of a Lie group G on a smooth manifold M is a smooth map $\cdot\colon G\times M \to M$ which is a group action.

DEFINITION 74. A *Lie subgroup* H of the Lie group G is a subgroup $H < G$ which is also a submanifold, in other words the image of an injective immersion of Lie groups.

EXAMPLE 75. Line of irrational slope on a torus.

REMARK 76. There is some play in the joints here.
(1) Enough to assume C^2, and may assume real-analytic (any C^2 structure is compatible with a unique smooth, even real-analytic, structure).
(2) Sophus Lie actually considered local Lie group actions.

2.3. Lie Algebras and the exponential map

2.3.1. Lie algebra. The Lie group G acts on itself by left multiplication. This *regular action* is a smooth action. In particular each $g \in G$ acts by translation on the set of vector fields of G, and we call a vector field X *left-invariant* if $g \cdot X = X$. Recall that for any manifold we have a surjective map $\{\mathcal{D}_M\} \to T_pM$.

LEMMA 77. *Restricting this map to the left-invariant vector fields on G gives a linear isomorphism* $\{$*left-invariant vector fields on G*$\} \to T_eG$.

PROOF. For the inverse map, for any manifold M a smooth action of G on M extends to a smooth action on TM by $g\cdot(p,v) = (gp, dg(v))$ where dg is the derivative of the map $g\cdot\colon M \to M$. In particular, G acts on TG . Now for $v \in T_eG$ the orbit $g \mapsto g\cdot(e,v)$ is a smooth left-invariant vector field. □

Note that if X,Y are left-invariant so is $[X,Y]$.

DEFINITION 78. The *Lie algebra* of G is the Lie algebra of left-invariant vector fields, equivalently the same Lie algebra realized as the tangent space T_eG. We write $\mathfrak{g} = \mathrm{Lie}(G)$ for the Lie algebra.

THEOREM 79. *If* $f \in \mathrm{Hom}(G,H)$ *then* $df\colon \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism.*

- For the converse see XXXX below.

LEMMA 80. *A connected topological group is generated by any open subset*

THEOREM 81. *Every subalgebra exponentiates to a subgroup*

PROOF. The distribution defined by the subalgebra is integrable, so apply Frobenius. The leaf through the origin is self-invariant, hence a subgroup. □

2.3.2. Exponential map.

LEMMA-DEFINITION 82. *The integral curves of left-invariant vector fields live forever. Write the integral curve of* $X \in \mathfrak{g}$ *as* $t \mapsto e_X(t) = \exp(tX)$. *We then have* $\exp((t+s)X) = \exp(tX)\exp(sX)$.

PROOF. By the Picard Theorem there is an integral curve $e_X(t)$ on some interval $(-\varepsilon, \varepsilon)$ with $e_X(0) = e$; it satisfies $\frac{d}{dx}e_X(t) = e_X(t)_* X$. Now let $\alpha(s)$ be any integral curve of this vector field, defined on some open interval (a,b). For any $s_0 \in (a,b)$ consider the curve

$$\tilde{\alpha}(s) = \alpha(s_0)e_X(s-s_0)\,.$$

We have $\tilde{\alpha}(s_0) = \alpha(s_0)$ and

$$\frac{d}{ds}\tilde{\alpha}(s) = \alpha(s_0)_* e_X(s-s_0)_* X = \tilde{\alpha}(s)_* X$$

hence

$$\frac{d}{ds}\tilde{\alpha}(s)\big|_{s=s_0} = \alpha(s_0)_* X\,.$$

It follows that both $\alpha(s)$, $\tilde{\alpha}(s)$ are integral curves of the vector field through $\alpha(s_0)$. By uniqueness the solution extends at least to $(a,b) \cup (s_0 - \varepsilon, s_0 + \varepsilon)$, and taking s_0 close to a,b we get that the solutions live forever. Applying the reasoning in particular to $\alpha(s) = e_X(s)$ we get that $e_X(s) = e_X(s_0)e_X(s-s_0)$ that is $e_X(s+t) = e_X(s)e_X(t)$.

Finally, for fixed a the curve $e_X(at)$ satisfies

$$\frac{d}{dt}e_X(at) = ae_X(at)_* X = e_X(at)_*(aX)\,,$$

so $e_X(at)$ is an integral curve of the field aX. Since $e_X(a0) = e_{aX}(0)$ as well we concldue that $e_X(at) = e_{aX}(t)$ and in particular that $e_X(t)$ only depends on the product tX and not on t, X separately, justifying the notation $\exp(tX)$. □

THEOREM 83. $\exp\colon \mathfrak{g} \to G$ *is a local diffeomorphism with derivative* Id.

PROOF. Solutions to ODE are differentiable wrt parameters, so $\exp(X) = e_X(1)$ is differentiable wrt X. To determine $d\exp$ we evaluate $\frac{d}{dt}|_{t=0}\exp(tX)$ in two different ways. On the one hand by the chain rule

$$\frac{d}{dt}|_{t=0}\exp(tX) = d\exp\left(\frac{d}{dt}(t \mapsto tX)\right) = (d\exp)(X)$$

and on the other hand by definition we have

$$\frac{d}{dt}|_{t=0}\exp(tX) = \frac{d}{dt}|_{t=0}e_X(t) = X\,.$$

□

- Interpretation: for each $X \in \mathfrak{g}$ we have a unique Lie group homomorphism $e_X : \mathbb{R} \to G$ such that $(de_X)(0) = X$.

COROLLARY 84. *Exponential map of* $\mathrm{GL}_n(\mathbb{R})$ *is given by the matrix exponential.*

PROOF. This is a Lie group homomorphism. □

COROLLARY 85 (Exponential coordinates). *For any direct sum decomposition* $\mathfrak{g} = \bigoplus_{i=1}^r V_i$ *the map* $(X_i)_{i=1}^r \mapsto \prod_{i=1}^r \exp(X_i)$ *is a local diffeomorphism.*

LEMMA 86. *Homomorphisms repsect the exponential map*

PROOF. Let $f\colon G \to H$. Then $f(\exp_G(tX))$ is a Lie group homomorphism $\mathbb{R} \to H$ with

$$\frac{d}{dt}|_{t=0}\left(f\left(\exp_G(tX)\right)\right) = df\left(\frac{d}{dt}|_{t=0}\exp_G(tX)\right) = df(X)$$

so

$$f\left(\exp_G(tX)\right) = \exp_H\left(t \cdot df(X)\right).$$

□

2.4. Closed Subgroups

2.4.1. Cartan's Theorem. Fix a Lie group G and let H be a closed subgroup. We want to show that H is a Lie subgroup. Our strategy is to identify the Lie algebra of H and then show that the restriction of (inverse of) the exponential map gives a coordinate patch.

For this fix small neighbourhoods of the identity $0 \in V_0 \subset \mathfrak{g}$ and $e \in U_0 \subset G$ such that $\exp\colon V_0 \to U_0$ is a diffeomorphism, and let $\log\colon U_0 \to V_0$ be its inverse. Also fix a norm $|\cdot|$ on $\mathfrak{g}$ (arbitraily).

Step 1: we identify the Lie algebra of H. For this let

$$\mathfrak{h}_1 = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp(tX) \in H\}$$
$$\mathfrak{h}_2 = \mathbb{R} \cdot \left\{\lim_{n\to\infty} \frac{\log h_n}{|\log h_n|} \mid \{h_n\}_{n=1}^{\infty} \subset H \setminus \{e\} : \lim_{n\to\infty} h_n = e\right\}.$$

LEMMA 87. *We have $\mathfrak{h}_1 = \mathfrak{h}_2$, which is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$.*

PROOF. If $X \in \mathfrak{h}_1$ is nonzero then $\lim_{t\to 0}\exp(tX) = e$ and

$$\lim_{t\to 0} \frac{\log\left(\exp(tX)\right)}{|\log\left(\exp(tX)\right)|} = \lim_{t\to 0}\frac{tX}{|tX|} = \frac{X}{|X|}.$$

It follows that $\frac{X}{|X|} \in \mathfrak{h}_2$ so $X \in \mathfrak{h}_2$ as well and $\mathfrak{h}_1 \subset \mathfrak{h}_2$. Conversely suppose $h_n \to e$ such that $\frac{\log h_n}{|\log h_n|} \to X$. Then for any sequence $m_n \in \mathbb{Z}$ we have

$$H \ni h_n^{m_n} = \exp\left(m_n \log h_n\right) = \exp\left(\frac{\log h_n}{|\log h_n|} \cdot m_n |\log h_n|\right).$$

Since $|\log h_n|$ are nonzero but tend to zero for any $t \in \mathbb{R}$ we can choose m_n so that $m_n|\log h_n| \to t$. Since exp is continuous it will follow that $h_n^{m_n} \to \exp(tX)$ and since H is closed it will follow that $\exp(tX) \in H$ and hence that $X \in \mathfrak{h}_1$.

Finally given $X, Y \in \mathfrak{h}_1$ with $X + Y \neq 0$ let $Z(t) = \log\left(\exp(tX)\exp(tY)\right)$. Then $Z(0) = 0$ and Then $\frac{d}{dt}|_{t=0}Z(t) = X + Y$ so Taylor expansion gives.

$$\log\left(\exp(tX)\exp(tY)\right) = t\left(X+Y\right) + O(t^2).$$

T We then have $|Z(t)| = |t|\,|X+Y| + O(t^2)$. It follows that

$$\frac{Z(t)}{|Z(t)|} \xrightarrow[t\to 0]{} \frac{X+Y}{|X+Y|}$$

and since $\exp(tX)\exp(tY) \in H$ for all t we get that $\frac{X+Y}{|X+Y|} \in \mathfrak{h}_2$ so $X + Y \in \mathfrak{h}_1$. This set is also clearly closed under rescaling so is a subspace.

Step 2: $\exp\colon \mathfrak{h} \to H$ is a local bijection.

For this fix a complementary subspace $\mathfrak{k}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, and let $V_1 \subset V_0$ be a small enough neighbourhood so that

$$\mathfrak{h} \oplus \mathfrak{k} \ni X + Y \mapsto \exp(X)\exp(Y)$$

is a diffeomorphism onto a neighbourhood $U_1 \subset U_0$. $\square$

LEMMA 88. *There is a neighbourhood $e \in U_2 \subset U_1$ such that* $\log(H \cap U_2) \subset \mathfrak{h}$ *and hence* $\log(H \cap U_2) = \mathfrak{h} \cap \log(U_2)$.

PROOF. If the claim fails we can find for each neighbourhood $U \subset U_1$ an element $h \in H \cap U$ such that $\log(h_n) \notin \mathfrak{h}$, and hence a sequence $h_n \to e$ with this property. Writing $h_n = \exp(X_n)\exp(Y_n)$ for $X_n \in \mathfrak{h}$, $Y_n \in \mathfrak{k}$ we have $X_n, Y_n \to 0$. Since X_n, Y_n are unique we have $Y_n \neq 0$ (otherwise we'd have $h_n = \exp(X_n)$ so $\log h_n \in \mathfrak{h}$). Now $\exp(Y_n) = \exp(-X_n)h_n \in H$ and taking logarithms we get that

$$\frac{Y_n}{|Y_n|} = \frac{\log(\exp(-X_n)h_n)}{|\log(\exp(-X_n)h_n)|}.$$

These belong to the unit sphere in $\mathfrak{k}$ which is compact, so let Y be any subsequential limit. Then $Y \in \mathfrak{h}_2 = \mathfrak{h}$ by construction, but also $Y \in \mathfrak{k}$ and the subspaces are disjoint, a contradition.

Now for any neighbourhood U_2 if $\log(H \cap U_2) \subset \mathfrak{h}$ then clearly $\log(H \cap U_2) \subset \mathfrak{h} \cap \log(U_2)$, but conversely if $X \in \mathfrak{h} \cap \log(U_2)$ then $\exp(X) \in H \cap U_2$ so $\mathfrak{h} \cap \log(U_2) \subset \log(H \cap U_2)$ and we get equality. $\square$

- It follows that $\mathfrak{h} = \mathbb{R}\log(H \cap U_2)$ since $\mathfrak{h} \cap \log(U_2)$ is open in $\mathfrak{h}$, giving us a third reconstruction of $\mathfrak{h}$.

Step 3: H is a submanifold with tangent space $\mathfrak{h}$.

THEOREM 89 (Cartan 1930). *Let $H < G$ be a closed subgroup. Then H is a submanifold of G, hence a Lie subgroup.*

PROOF. By continuity of the multiplication on G there is a neighbourhood $e \in U_3$ such that $U_3^{-1}U_3U_3 \subset U_2$ (in particular $U_3 \subset U_2$). Let $V_3 = \log(U_3 \cap H)$ and for $h \in H$ let $U_h = h(U_3 \cap H)$ and let $\varphi_h \colon U_h \to \mathfrak{h}$ be given by $\varphi_h(x) = \log(h^{-1}x)$. This is an atlas on H: suppose that $x \in U_{h_1} \cap U_{h_2}$. The inverse of the first chart is the map $V_3 \ni X \mapsto h_1 \exp(X)$ and applying the first chart we get

$$\begin{aligned}\left(\varphi_{h_2} \circ \varphi_{h_1}^{-1}\right)(X) &= \log\left(h_2^{-1}h_1\exp(X)\right) \\ &= \log\left(((x^{-1}h_2)^{-1}\left(x^{-1}h_1\right)\exp(X)\right).\end{aligned}$$

Now since $x \in U_{h_1} \cap U_{h_2}$ we have $x^{-1}h_1, x^{-1}h_2 \in U_3$ so $\left(((x^{-1}h_2)^{-1}\left(x^{-1}h_1\right)\exp(X)\right) \in U_3^{-1}U_3U_3 \subset U_2$ and we conclude that $\varphi_{h_2} \circ \varphi_{h_1}^{-1} \colon \mathfrak{h} \to \mathfrak{h}$ is smooth. $\square$

REMARK 90. By the construction of $\mathfrak{h}_1$ we see that $\mathfrak{h}$ is the Lie algebra of H, in particular a Lie subalgebra of G.

COROLLARY 91. *Let $f \colon G \to H$ be a* continuous *homomorphism of Lie groups. Then f is smooth. ("The category of Lie groups is a full subcategory of the category of topological groups")*

PROOF. Let $\Gamma_f = \{(g, f(g)) \mid g \in G\} \subset G \times H$ be the graph of f. Then Γ_f is the image of a group homomorphism $G \to G \times H$ hence an abstract subgroup. It is closed by the continuity of f: if $(g_n, f(g_n)) \to (g, h)$ in $G \times H$ then $g_n \to g$ and therefore $f(g_n) \to f(g)$ so $h = f(g)$ so $(g, h) \in \Gamma_f$. By Cartan's Theorem 89 Γ_f is a smooth submanifold, and it follows that f is smooth. $\square$

The argument of the above theorem also gives the following

THEOREM 92. *Let H be a closed connected subgroup of G. Then $H\ G$ has a unique manifold structure such that $\pi\colon H\backslash G \to G$ is smooth. Furthermore, the regular action of G on $H\backslash G$ is a Lie group action.*

PROOF. Decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ as before, giving us maps $\log_{\mathfrak{h}}\colon V_1 \to \mathfrak{h}$, $\log_{\mathfrak{k}}\colon V_1 \to \mathfrak{k}$ inverse to $X + Y \mapsto \exp(X)\exp(Y)$. Choose U such that $U^{-1}UU \subset U_1$. For $g \in G$ define a chart $(\pi(U)g, \varphi_g)$ by $\varphi_g(Hx) = \log_{\mathfrak{k}}(xg^{-1})$ ... □

2.4.2. Topological applications: covering groups.

LEMMA 93. *A Lie group homomorphism is a covering iff its derivative is an isomorphism*

PROOF. A covering map is a local diffeo, hence gives isom of Lie algebras. Conversely suppose $d_e f$ is isomorphism. By homogeneity df is injective at each point so f is a local diffeomorphism. The kernel $K = \mathrm{Ker}(f)$ is a closed subgroup which is zero-dimensional (lie algebra $\{0\}$!) hence discrete. Let $U \subset G$ be a small enough neighbourhood so that its translates by K are disjoint and such that $f\restriction_U$ is a diffeo. Then $f^{-1}(f(U)) \simeq K \times U$ and the map is a cover. □

THEOREM 94. *Let $df\colon \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected and H is connected then this lifts to $f \in \mathrm{Hom}(G,H)$.*

PROOF. Let $\Gamma_{df} = \{(X, df(X)) : X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of df, which is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h} = \mathrm{Lie}(G \times H)$. Let $\Gamma_f \subset G \times H$ be the corresponding Lie subgroup. The projections $\pi_1\colon G \times H \to G$, $\pi_2\colon G \times H \to H$ are Lie group homomorphism, and set $\pi = \pi_1\restriction_{\Gamma_f}$. Then $d\pi = (d\pi_1)\restriction_{\Gamma_{df}}\colon \Gamma_{df} \to \mathfrak{g}$ is an isomorphism, so $\pi\colon \Gamma_f \to G$ is a covering map. Since G is simply connected π is an isomorphism, and $\pi_2 \circ \pi^{-1}\colon G \to H$ is then Lie group homomorphism with graph Γ_f and hence differential $d\pi$. □

We note the following without proof.

THEOREM 95 (Ado). *Every finite-dimensional Lie algebra has a faithful representation into $\mathfrak{gl}_n(\mathbb{R})$.*

COROLLARY 96. *Every Lie algebra is the Lie algebra of some group.*

EXERCISE 97. Let $\mathcal{S}(G)$ be the lattice of closed subgroups of a compact Lie group G. Then the map $\mathcal{S}(G) \to \omega \times \omega$ given by $H \mapsto (\dim H, \#\pi_0(H))$ is order-preserving for the lexicographic order. Conclude that there is no infinite descending chain of closed subgroups of G. Give a counterexample when G is non-compact.

2.5. The adjoint representation

EXAMPLE 98. Let $(R_t f)(x) = f(x+t)$ be the translation operator for functions on $\mathbb{R}$. Then Taylor expansion can be viewed as $R_t f = \sum_{k=0}^{\infty} \frac{t^k}{k!}\frac{d^k f}{dx^k} = \exp\left(t\frac{d}{dx}\right) \cdot f$.

LEMMA 99 (Infinitesimal translation). *The operator $R_{\exp(tX)}$ on $C^\infty(G)$ has the Taylor expansion $\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$ in the sense that for all $N \geq 0$ and compact $\Omega \subset G$ we have for $g \in \Omega$*

$$f(g\exp(tX)) = \left(R_{\exp(tX)}(f)\right)(g) = \sum_{k=0}^{N} \frac{t^k}{k!}\left(X^k f\right)(g) + O_{f,\Omega}\left(|tX|^{k+1}\right)$$

PROOF. Since both $R_{\exp(tX)}$ and the operators X^k commute with left translations it's enough to prove this at $g=1$. This is the Taylor expansion with remainder of the smooth function $t \mapsto f(\exp(tX))$ using the fact that $\frac{d}{dt} f(\exp(tX)) = (Xf)(\exp(tX))$ which we can iterate. $\square$

REMARK 100. A Lie group has a unique real analytic structure and then for $f \in C^\omega(G)$ we have $f(g\exp(tX)) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(X^k f\right)(g)$.

COROLLARY 101. *In the same sense we have* $\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{1}{2}t^2[X,Y] + O(t^3)\right)$.

PROOF. $\log(\exp(tX)\exp(tY))$ is a smooth function $(-\varepsilon,\varepsilon) \to \mathfrak{g}$ and we have

$$\begin{aligned}
R_{\exp(tX)}R_{\exp(tY)} &= \left(1 + tX + \frac{1}{2}t^2X^2 + O(t^3)\right)\left(1 + tY + \frac{1}{2}t^2Y^2 + O(t^3)\right) \\
&= 1 + t(X+Y) + \frac{1}{2}t^2\left(X^2 + Y^2 + XY\right) + O(t^3) \\
&= 1 + t(X+Y) + \frac{1}{2}t^2(X+Y)^2 + \frac{1}{2}t^2(XY - YX) + O(t^3) \\
&= 1 + t\left(X + Y + \frac{1}{2}t[X,Y]\right) + \frac{1}{2}t^2\left(X + Y + \frac{1}{2}t[X,Y]\right)^2 + O(t^3).
\end{aligned}$$

$\square$

OBSERVATION 102. *We see that in exponential coordinates the Lie bracket exactly gives the deviation of the multiplication law of G from the additive law of* $\mathfrak{g}$ *(and fully determines the rest of the Taylor expansion!)*

COROLLARY 103. *Simlarly we have*

$$\begin{aligned}
\exp(tX)\exp(sY)\exp(-tX) &= \left(1 + tX + \frac{1}{2}t^2X^2 + O(t^3)\right)\left(1 + sY + \frac{1}{2}s^2Y^2 + O(s^3)\right)\left(1 - tX + \frac{1}{2}t^2X^2 + O(t^3)\right) \\
&= 1 + tY + tsXY - stYX - t^2X^2 + t^2X^2 + \frac{1}{2}s^2Y^2 + O(s^3, t^3, s^2t, t^2s) \\
&= 1 + sY + ts[X,Y] + \frac{1}{2}s^2Y^2 + O(s^3, t^3, s^2t, t^2s).
\end{aligned}$$

DEFINITION 104. Let $g \in G$. Then $\mathrm{Ad}_g \colon G \to G$ given by $\mathrm{Ad}_g(x) = gxg^{-1}$ is an automorphism, in particular a group homomorphism. We also write Ad_g for its derivative, $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$.

LEMMA 105. $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ *is a smooth representation.*

DEFINITION 106. Write $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ for the derivative of the adjoint representation.

THEOREM 107. $\mathrm{ad}_X \cdot Y = [X,Y]$.

PROOF. By Corollary 103we have

$$\mathrm{Ad}_{\exp(tX)} \exp(sY) =$$

More precisely, this means that for $f \in C^\infty(G)$,

$$\langle df_e, \mathrm{ad}_X \cdot Y \rangle = \frac{d}{ds}\Big|_{s=0} \left\langle df_{\exp(sY)}, X_e \cdot (\exp(sY))_* - X_{\exp(sY)} \right\rangle .$$

Now

$$\frac{d}{ds}\Big|_{s=0}\left\langle df_{\exp(sY)}, X_e\cdot(\exp(sY))_*\right\rangle = \frac{d}{ds}\Big|_{s=0}\left\langle d(R_{\exp(sY)}f)_e, X_e\right\rangle = \left\langle \frac{d}{ds}\Big|_{s=0} d\left(g\mapsto f(g\exp(sY))\right)_e, X_e\right\rangle = (XYf)($$

and

$$\frac{d}{ds}\Big|_{s=0}\left\langle df_{\exp(sY)}, X_{\exp(sY)}\right\rangle = (YXf)(e)$$

so we are done. □

COROLLARY 108. $\mathrm{ad}\colon \mathfrak{g}\to\mathrm{End}(\mathfrak{g})$ *is a Lie algebra representation:* $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

PROOF. This follows immediately from the Jacobi identity. □

COROLLARY 109. *Let $H < G$ be connected Lie groups. Then H is normal iff $\mathfrak{h}$ is a Lie ideal.*

PROOF. If H is normal then H is Ad-stable hence $\mathfrak{h}$ is Ad-stable hence $\mathfrak{h}$ is ad-stable. Conversely, for X close enough to the origin we have $\exp(\mathrm{ad}_X) = \sum_{k=0}^{\infty}\frac{1}{k!}(\mathrm{ad}_X)^k$. Now if $\mathfrak{h}$ is ad_X-stable it follows that it is also $\exp(\mathrm{ad}_X)$-stable and hence $\mathrm{Ad}_{\exp X}$-stable. But by the group-algebra correspondence this means H is $\mathrm{Ad}_{\exp X}$-stable. Since the small X generate G we are done. □

COROLLARY 110. *Let G be connected. Then* $Z(G) = \ker(\mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g}))$.

PROOF. $g\in G$ is central iff for all small enough X, $g\exp X g^{-1} = \exp X$ iff $\exp(\mathrm{Ad}_g X) = \exp X$ iff $\mathrm{Ad}_g X = X$. □

COROLLARY 111. *Let G be connected. Then $\mathfrak{g}$ is abelian iff G is abelian iff* $\exp\colon \mathfrak{g}\to G$ *is a surjective group homomorphism.*

PROOF. If $\mathrm{ad}_X = 0$ for all X then $\exp(\mathrm{ad}_X) = \mathrm{Id}$ for all X so a neighbourhood of the identity is contained in Ker(Ad). If G is abelian let $X, Y\in\mathfrak{g}$. Then $t\mapsto \exp(tX)\exp(tY)$ is a group homomorphism $\mathbb{R}\to G$. Since its derivative at $t=0$ is $X+Y$ we conclude that $\exp(tX)\exp(tY) = \exp(t(X+Y))$. Now setting $t=1$ shows that exp is a homomorphism, and since the image contains a generating set it's surjective. Finally, if exp is a surjective homomorphism then its image G is abelian. □

THEOREM 112. *A connected abelian Lie group is of the form $\mathbb{R}^a\times\mathbb{T}^b$. Its exponential map is a covering map.*

PROOF. Ker(exp) is a discrete subgroup of $\mathbb{R}^n$. □

EXERCISE 113. A compact abelian Lie group has the form $\mathbb{T}^b\times A$ where A is a finite abelian group.

CHAPTER 3

Compact Lie groups

3.1. Linearity

As an application of our representation theory of compact groups we get:

THEOREM 114. *Every compact Lie group has a faithful finite-dimensional representation. Equivalently, every compact group is isomorphic to a closed subgroup of some $U(n)$.*

PROOF. The representation of G on $L^2(G)$ is faithful. By Peter–Weyl it follows that $\bigcap_{\pi\in\hat{G}}\mathrm{Ker}(\pi) = \{e\}$, and since there is no infinite descending sequence of closed subgroups finitely many irreducibles suffice. □

REMARK 115. Closed linear groups are, in fact, algebraic (i.e. the zero set of the polynomials that vanish on them), as are the continuous homomorphisms between them.

3.2. Characters and cocharacters of tori

Let $\mathbb{R}^n/\mathbb{Z}^n$, $\mathbb{R}^m/\mathbb{Z}^m$ be tori. We'd like to study $\mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$. The cases $m = 1$ (characters) and $m = n$ (automorphisms) are particularly important.

First, let $f\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism. Extending scalars gives a homomorphism $f_{\mathbb{R}} = f \otimes_{\mathbb{Z}} \mathbb{1}\colon \mathbb{Z}^n \otimes \mathbb{R} \to \mathbb{Z}^m \otimes \mathbb{R}$. Since $f_{\mathbb{R}}(\mathbb{Z}^n) = f(\mathbb{Z}^n) \subset \mathbb{Z}^m$, $f_{\mathbb{R}}$ descends to a homomorphism $\bar{f}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^m/\mathbb{Z}^m$.

LEMMA 116. *The map $f \mapsto \bar{f}$ is an isomorphism* $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m) \to \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$.

PROOF. We need to construct the inverse map. For this let $\exp_n\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the quotient map, which is also the exponential map of this commutative Lie group with kernel $\mathbb{Z}^n$. Then given $\bar{f} \in \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$ consider the linear map $d\bar{f} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. The usual identity $\bar{f}(\exp_n X) = \exp_m\left(d\bar{f}(X)\right)$ here reads $\exp_m \circ df = \bar{f} \circ \exp_n$, in other words that $d\bar{f}(\mathbb{Z}^n) \subset \mathbb{Z}^m$. The desired element of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$ is them $f = d\bar{f}\restriction_{\mathbb{Z}^n}$. □

COROLLARY 117. $\mathrm{Aut}(\mathbb{T}^n) \simeq M_n(\mathbb{Z})^\times = \mathrm{GL}_n(\mathbb{Z})$. *In particular,* $\mathrm{Aut}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$ *is discrete.*

COROLLARY 118. $\widehat{\mathbb{T}^n} = \mathrm{Hom}\left(\mathbb{T}^n, S^1\right) = \{e(\underline{k}\cdot\underline{x})\}_{\underline{k}\in\hat{\mathbb{Z}}^n}$ *where* $\hat{\mathbb{Z}}^n = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ *is the dual lattice, and $e(z) = e^{2\pi i z}$.*

PROOF. $z \mapsto e(z)$ is an isomorphism $\mathbb{R}/\mathbb{Z} \to S^1$. □

LEMMA 119. *Tori are topologically generated by single elements.*

PROOF. Let $\{1\} \cup \{\xi_i\}_{i=1}^n \subset \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then $\underline{\xi}$ is such an element. In fact (Weyl equidistribution) every orbit $\left\{\underline{x} + j\underline{\xi}\right\}_{j=1}^\infty$ is equidistributed in the torus. □

COROLLARY 120. *$\mathbb{T}^n \times C_m$ is also topologically generated by a single element.*

PROOF. Let $\underline{\xi}$ be an irrational element as above, and let $g \in C_m$ be a generator. Then $\left(\underline{\xi}, g\right)$ is a generator. $\square$

3.3. The exponential map of a compact group

From now on let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. Since G is compact we may fix a G-invariant inner product (and associated Euclidean norm) on $\mathfrak{g}$.

LEMMA 121. *A connected compact Lie group has a bi-invariant Riemannian metric*

REMARK 122. The map $g \mapsto g^{-1}$ is an isometry of this metric. In other words, we have a *symmetric space*. (c.f. PS ??)

PROPOSITION 123. *Fix a bi-invariant metric on G. Then the Riemannian and Lie exponential maps agree.*

PROOF. Let $\gamma(t)$ be a Riemannian geodesic based at the origin. Then $t \mapsto \gamma(t_0 + t)$, $t \mapsto \gamma(t_0)\gamma(t)$ and $t \mapsto \gamma(t)\gamma(t_0)$ are also geodesics (because the group acts by isometries) which meet at $t = 0$ and have the same derivative at that time. It follows that $\gamma(t_0+t) = \gamma(t_0)\gamma(t)$, that is that the geodesic is a one-parameter subgroup. $\square$

COROLLARY 124. *The exponential map of a connected compact Lie group is surjective.*

3.4. Maximal Tori

Fix a compact connected Lie group G.

3.4.1. Tori. A *torus* in G is a subgroup T of G isomorphic to $\mathbb{T}^n$ for some n, equivalently a compact connected commutative subgroup. A *maximal torus* is a torus not properly contained in another torus, equivalenty a maximal connected commutative subgroup (taking the closure shows that such a subgroup is necessarily compact). If $T \subset T'$ are distinct tori then by connectedness $\dim T < \dim T' \leq \dim G$, so each torus is contained in a maximal torus.

LEMMA 125. *Every $g \in G$ is contained in a torus, hence in a maximal torus.*

PROOF. Suppose $g = \exp(X)$ for some non-zero $X \in \mathfrak{g}$. Then $\{\exp(tX)\}_{t\in\mathbb{R}}$ is a connected commutative subgroup of G. Its closure is still connected and commutative, hence a torus. $\square$

LEMMA 126. *Let T be a torus in G, and let $\mathfrak{t}$ be its Lie algebra. Then:*

(1) *$Z_G(T)$ is connected.*
(2) $Z_G(\mathfrak{t}) = Z_G(T)$
(3) $\mathrm{Lie}\, Z_G(T) = Z_{\mathfrak{g}}(\mathfrak{t})$.
(4) $N_G(T)^\circ = Z_G(T)$.

PROOF.

(1) Let $g \in Z_G(T)$ and let $S = \overline{\langle g, T\rangle}$ be the closed subgroup generated by g, T. Then S is a closed commutative subgroup, so its connected component is a torus $S^0 \supset T$. The image of g is a topological generator of S/T hence of the finite group S/S^0, so this group is cyclic and $S \simeq \mathbb{T}^{\dim S} \times S/S^0$ is topologically generated by a single element h, and any

torus containing h contains S. It follows that $Z_G(T)$ is the union of the tori containing T and hence is connected.

(2) If $g \in Z_G(T)$ then $\mathrm{Ad}_g \in \mathrm{Aut}(T)$ being trivial means that $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{t})$ is trivial. Conversely, the exponential map of T is surjective and for any $H \in \mathfrak{t}$ and $g \in Z_G(\mathfrak{t})$ we have
$$\mathrm{Ad}_g(\exp H) = \exp(\mathrm{Ad}_g H) = \exp H .$$

(3) If $X \in Z_{\mathfrak{g}}(\mathfrak{t})$ then for any $s \in \mathbb{R}$, $\mathrm{Ad}_{\exp(sX)} \restriction_{\mathfrak{t}} = \exp(\mathrm{ad}_{sX} \restriction_{\mathfrak{t}}) = \exp(0) = \mathrm{Id}$ and hence $\exp(sX) \in Z_G(\mathfrak{t})$ and $X \in \mathrm{Lie}(Z_G(\mathfrak{t}))$. Conversely, suppose that $\mathrm{Ad}_{\exp(sX)} \in Z_G(\mathfrak{t})$ for all s. Differentiating with respect to s we get that $\mathrm{ad}_X \restriction_{\mathfrak{t}} = 0$ that is that $X \in Z_{\mathfrak{g}}(\mathfrak{t})$.

(4) Finally, let $N_G(T)$ act on T by conjugation. This gives a continuous homomorphism $N_G(T) \to \mathrm{Aut}(T) \simeq \mathrm{GL}_r(\mathbb{Z})$. Since the latter group is discrete, the connected component is in the kernel and hence $N_G(T)^\circ \subset Z_G(T)$. Since $Z_G(T) \subset N_G(T)$ is connected we also have the reverse inclusion.

□

3.4.2. Weyl groups. Now let T be a maximal torus of G.

LEMMA 127. *We have $N_G(T)^\circ = Z_G(T) = T$*

PROOF. $Z_G(T)$ is connected. Then any $g \in Z_G(T)$ belongs to some torus $S \subset Z_G(T)$. If $g \notin T$ then ST would be a torus properly containing T. □

DEFINITION 128. The *analytic Weyl group* (or just "Weyl group") of G is $W(G:T) \stackrel{\mathrm{def}}{=} N_G(T)/Z_G(T) = N_G(T)/T$.

THEOREM 129. *All maximal tori of G are conjugate.*

PROOF. Let S,T be maximal tori and let $X \in \mathrm{Lie}\,S$, $Y \in \mathrm{Lie}\,T$ be generic elements (that is $\exp X, \exp Y$ are topological generators of the respective groups). Equip $\mathfrak{g} = \mathrm{Lie}\,G$ with a G-invariant inner product, and let $g \in G$ minimize
$$f(g) = \|\mathrm{Ad}(g)X - Y\|^2 .$$
Expressing f as:
$$\begin{aligned} f(g) &= \|\mathrm{Ad}(g)X\|^2 + \|Y\|^2 - 2\langle \mathrm{Ad}(g)X, Y\rangle \\ &= \|X\|^2 + \|Y\|^2 - 2\langle \mathrm{Ad}(g)X, Y\rangle \end{aligned}$$
we see that we are minimizing $\langle \mathrm{Ad}(g)X, Y\rangle$. Suppose the minimum is at g_0, and consider the derivative there. For every $Z \in \mathfrak{g}$ the derivative in the direction Z is:
$$\langle \mathrm{ad}\,Z \cdot (\mathrm{Ad}(g_0)X), Y\rangle .$$
Letting $X_0 = \mathrm{Ad}(g_0)X$ we see that
$$\begin{aligned} 0 &= \langle \mathrm{ad}\,Z \cdot (\mathrm{Ad}(g_0)X), Y\rangle \\ &= \langle [Z, X_0], Y\rangle = -\langle [X_0, Z], Y\rangle \\ &= -\langle \mathrm{ad}\,X_0 \cdot Z, Y\rangle \\ &= \langle Z, \mathrm{ad}\,X_0 \cdot Y\rangle = \langle Z, [X_0, Y]\rangle \end{aligned}$$

where we use that in every unitary representation π, $d\pi(X)$ is anti-hermitian. Since Z is arbitrary, we see that $[X_0, Y] = 0$. This means that $X_0 \in Z_{\mathfrak{g}}(Y) = \mathfrak{t}$. But since X_0 is generic for $g_0 S g_0^{-1}$ we conclude that $g_0 S g_0^{-1} = T$. □

LEMMA 130. *Let the compact group G act on the Hausdorff space X. Then*

(1) *Every open neighbourhood of an orbit contains a G-invariant open neighbourhood.*
(2) *The quotient $G\backslash X$ is Hausdorff.*

PROOF. Given an open subset $U \subset X$ containing an orbit Gx let $V = \{y \in G \mid Gy \subset U\}$, which is clearly G-invariant, contained in U, and contains Gx. To see that V is open let $\{z_n\}_{n\in N}$ be a net in the complement $X \setminus V$ converging to $z \in X$. By hypothesis for each z_n we have $g_n \in G$ such that $g_n z_n \notin U$. Since G is compact there is a subnet $M \subset N$ such that $\{g_n\}_{n\in M}$ converges to $g \in G$. By the continuity of the group action the net $\{g_n z_n\}_{n\in M}$ converges to gz and since U is closed we have $gz \notin U$ so $z \notin V$. It follows that $X \setminus V$ is closed, so V is open.

Now let $x, y \in X$ have disjoint G-orbits. Since X is Hausdorff and the orbits Gx, Gy are compact there are disjoint neighbourhoods $Gx \subset U_x$, $Gy \subset U_y$. By the first part we have G-invariant neighbourhoods $Gx \subset V_x \subset U_x$ and $G_y \subset V_y \subset U_y$. The images of V_x, V_y in $G\backslash X$ are then open and disjoint, thus separate the images of x, y. □

COROLLARY 131. *We have a homomorphism $T/W = G/\operatorname{Ad}(G)$, hence an isomomorphism $C(T)^W = C(T/W) = C(G/\operatorname{Ad}(G)) = C(G)^{\operatorname{Ad} G}$.*

PROOF. The continuous inclusion $T \hookrightarrow G$ induces a continuous map $T/W \to G/\operatorname{Ad}(G)$, which is surjective since every $g \in G$ is contained in a maximal torus which is conjugate to T, so T contains representatives for all conjugacy classes.

To see that the map is injective let $t, t' \in T$ be conjugate in G, say $gt'g^{-1} = t$. Then $t \in gTg^{-1}$, and it follows that T and gTg^{-1} are maximal tori in $Z_G(t)$. We therefore have $z \in Z_G(t)^{\circ}$ such that $zgTg^{-1}z^{-1} = T$, that is $zg \in N_G(T)$. We also have $zgt'g^{-1}z^{-1} = ztz^{-1} = t$.

Finally the spaces T/W and $G/\operatorname{Ad}(G)$ are compact Hausdorff spaces as quotients of compact spaces, and Hausdorff by the Lemma. Thus the continuous bijection between them is a homeomorphism. □

REMARK 132. The situation for noncompact groups is much more complicated.

3.4.3. Example: three-dimensional groups. Let $G = \mathrm{SU}(2)$ act on $\mathbb{C}^2$. The action on S^3 is simply transitive, so $\mathrm{SU}(2) \simeq S^3$; in particular it is simply connected. Now $Z(\mathrm{SU}(2)) = \{\pm I\}$,so the groups it covers are $\mathrm{SU}(2)$ and its image by the adjoint representation.

LEMMA 133. *The maximal tori are the maximal subalgebras of* $\mathfrak{so}(3)$.

PROOF. Let $\mathfrak{t} = \operatorname{Span}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Then the action of $\mathfrak{t}$ on its orthogonal complement in $\mathfrak{so}(3)$ is irreducible. □

PROPOSITION 134. *Let G be a three-dimensional connected compact Lie group. Then either G is abelian or G covers* $\mathrm{SO}(3)$.

PROOF. Consider the adjoint representation $\operatorname{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. Choosing a G-invariant inner product on $\mathfrak{g}$, the image lies in its orthogonal group, in fact in $\mathrm{SO}(3)$ since G is connected. The image of this map is a closed subgroup of $\mathrm{SO}(3)$; if it is proper then by the Lemma it is either a one-dimensional torus or the trivial group. But since $\mathfrak{g}$ is non-commutative we cannot have $\mathfrak{g}/Z_{\mathfrak{g}} \simeq \mathbb{R}$ nor $\mathfrak{g}/Z_{\mathfrak{g}} \simeq \{0\}$. □

3.5. Roots and weights

3.5.1. Weights. Let T be a torus. Let (π, V) be a finite-dimensional representation of T on a complex vector space. By the theory for general compact groups we have a direct sum decomposition

$$V = \oplus_{\chi \in \hat{T}} V_\chi .$$

Since T is commutative, $\hat{T} = \mathrm{Hom}_{\mathrm{cts}}\left(T, S^1\right)$ and $V_\chi = \{\underline{v} \in V \mid \pi(t)\underline{v} = \chi(t)\underline{v}\}$. We call $\left\{\chi \in \hat{T} \mid V_\chi \neq \{0\}\right\}$ the *exponential weights* of V, V_χ the *weight spaces*.

We now find an alternative parametrization of $\hat{T}$. For this let $\mathfrak{t}$ be the Lie algebra, $\exp\colon \mathfrak{t} \to T$ the exponential map. We have seen that exp is also the universal covering map of T; we write Λ for its kernel and call it the *integral lattice*.

Identify the Lie algebra of S^1 with $\mathbb{R}$ so that the exponential map is $e(z) = e^{2\pi i z}$. For a character $\chi \in \hat{T}$ write $\alpha = d\chi \in \mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ for its derivative, giving the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{t} & \xrightarrow{\exp} & T \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \chi} \\ \mathbb{R} & \xrightarrow{e} & S^1 \end{array}$$

Now $\chi \circ \exp$ vanishes on Λ, and it follows that $\alpha(\Lambda) \subset \ker(e) = \mathbb{Z}$. The converse is also clear, so

CONCLUSION 135. $\chi \in \hat{T}$ iff $\alpha \in \Lambda^* = \{\nu \in \mathfrak{t}^* \mid \nu(\Lambda) \subset \mathbb{Z}\} \simeq \mathrm{Hom}(\Lambda, \mathbb{Z})$.

We call Λ^* the *weight lattice* of T, and from now on we index weight spaces with the weights $\alpha \in \Lambda^*$ rather than the corresponding exponential weights $\chi_\alpha \in \mathrm{Hom}\left(T, S^1\right)$. Explicitely given $\alpha \in \Lambda^*$ and $H \in \mathfrak{t}$ we have $\chi_\alpha(\exp H) = e^{2\pi i \alpha(H)}$.

3.5.2. Complexification. Suppose now that T acts on a *real* vector space V. Since every non-trivial character of T takes complex values, V realizes no character of T, and we consider the *complexification* $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} V$.

The complex conjugation operator $z \mapsto \bar{z}$ of $\mathbb{C}$ then extends to an operation $\underline{v} \mapsto \bar{\underline{v}}$ on $V_\mathbb{C}$ (fixing the image of V in $V_\mathbb{C}$), and also $\mathrm{End}_\mathbb{C}(V_\mathbb{C})$ (fixing the image of $\mathrm{End}_\mathbb{R}(V)$ there).

EXERCISE 136. A ($\mathbb{C}$-linear) subspace $W \subset V_\mathbb{C}$ is of the form $U_\mathbb{C}$ for an ($\mathbb{R}$-linear) subspace $U \subset V$ iff $W = \overline{W}$.

The T-action on V then extends to a T-action on $V_\mathbb{C}$, so we may write $V_\mathbb{C} = \bigoplus_{\alpha \in \Lambda^*} V_\alpha$. Then for any $H \in \mathfrak{t}$ and $\underline{v} \in V_\alpha$ we have

$$\pi(\exp(H)) \cdot \underline{v} = e^{2\pi i \alpha(H)} \underline{v} .$$

Taking complex conjugates it follows that

$$\pi(\exp(H)) \cdot \bar{\underline{v}} = e^{-2\pi i \alpha(H)} \bar{\underline{v}} ,$$

in other words that $\bar{\underline{v}} \in V_{-\alpha}$. We conclude that $\alpha \neq 0$ is a weight iff $-\alpha$ is a weight and that $\bar{V}_\alpha = V_{-\alpha}$.

3.5.3. Roots. Let G be a connechted compact Lie group and fix a maximal torus $T \subset G$.

DEFINITION 137. The *rank* of G is the integer $\operatorname{rk} G = \dim T$. The *semisimple rank* of G is the rank of $G/Z(G)$, in other words the integer $\dim T - \dim Z(G)$.

DEFINITION 138. The *real roots* of G (with respect to T) are the non-zero weights of the adjoint action of T on $\mathfrak{g}$. Write $\Phi = \Phi(G : T)$ for the set of roots.

The weight space $\mathfrak{g}_0$ corresponding to the weight 0 (that is, the space of T-fixed vectors) is self-conjugate, hence is the complexification of the space of T-fixed vectors in $\mathfrak{g}$. Since $Z_G(T) = T$ we see that this is exactly $\mathfrak{t}_{\mathbb{C}}$ so we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha .$$

REMARK 139. We will now eventually compute the structure of $\mathfrak{g}$ from this decomposition.

Let $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{g}_\alpha$. We then have

$$\operatorname{Ad}(\exp(tH)) \cdot X_\alpha = e^{2\pi i \alpha(H)} X_\alpha .$$

Differentiating with respect to t we conclude that

$$\operatorname{ad}_H \cdot X_\alpha = 2\pi i \alpha(H) X_\alpha .$$

In other words, $\mathfrak{g}_\alpha$ is a joint eigenspace of $\{\operatorname{ad}_H\}_{H \in \mathfrak{t}}$ where the eigenvalue of H is $2\pi i \alpha(H)$.

DEFINITION 140. Given a real root α, the map $H \mapsto 2\pi i \alpha(H)$ will be called the associated *complex root.* We denote both by α, but it should be clear from context which is intended. Note that the real root is an element of $\mathfrak{t}^*_{\mathbb{R}}$ while the latter is a purely imaginary element of $\mathfrak{t}^*_{\mathbb{C}}$. Generaly the real roots are useful when studying representation theory and the "root system". The complex roots are useful when studying structure theory, that is in computing commutators in $\mathfrak{g}$. Recall that we also have an associated *exponential root* $\chi_\alpha \colon T \to S^1$ such that $\operatorname{Ad}_t \cdot X_\alpha = \chi_\alpha(t) X_\alpha$ whenever $t \in T$, $X_\alpha \in \mathfrak{g}_\alpha$.

LEMMA 141. *For* $\alpha, \beta \in \Lambda^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

PROOF. Let $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$. Then by the Jacobi identity (writing α for the *complex* root)

$$\begin{aligned}
[H, [X_\alpha, X_\beta]] &= -[X_\alpha, [X_\beta, H]] - [X_\beta, [H, X_\alpha]] \\
&= -[X_\alpha, -\beta(H) X_\alpha] - [X_\beta, \alpha(H) X_\alpha] \\
&= (\beta(H) + \alpha(H)) [X_\alpha, X_\beta] \\
&= ((\alpha+\beta)(H)) [X_\alpha, X_\beta] .
\end{aligned}$$

□

We are now ready to begin studying structure theory in earnest. The following argument is taken from [**?**, Thm.\{} V.1.5]

THEOREM 142. *If* $\operatorname{rk} G = 1$ *then* G *is either* $\mathrm{SO}(3)$ *or* $\mathrm{SU}(2)$.

PROOF. We begin with two preliminary observations

(1) Given $\beta \in \Phi$ let $X_\beta \in \mathfrak{g}_\beta$. Then $X_{-\beta} = \bar{X}_\beta \in \mathfrak{g}_{-\beta}$ and we may consider $H_\beta = \left[X_\beta, X_{-\beta}\right]$. If H_β were zero $\mathrm{Span}\left\{X_\beta, X_{-\beta}\right\} \subset \mathfrak{g}_\mathbb{C}$ would be a two-dimensional commutative subalgebra of $\mathfrak{g}_\mathbb{C}$. Since this subspace is stable by complex conjugation it would be the complexification of a two-dimensional commutative subalgebra of $\mathfrak{g}$, and such subalgebras don't exist when $\mathrm{rk}\, G = 1$. It follows that $H_\beta \neq 0$ in such circumstances. We also note that $\bar{H}_\beta = \left[\bar{X}_\beta, \bar{X}_{-\beta}\right] = \left[X_{-\beta}, X_\beta\right] = -H_\beta$. It follows that $H_\beta \in i\mathfrak{t}_\mathbb{R}$, and that $iH_\beta \in \mathfrak{t}$.

(2) Since $\dim_\mathbb{R} \mathfrak{t} = 1$, for any non-zero $H \in \mathfrak{t}$ every real root α is determined by the non-zero real number $\alpha(H)$, and we *order* the roots by these numbers.

In particualr let β be the smallest positive root (with respect to some choice of H), choose X_β (arbitrarily in $\mathfrak{g}_\beta$) and $X_{-\beta}$ as above and let

$$V = \mathbb{C}X_{-\beta} \oplus \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha .$$

We then have:

(1) V is ad_{X_β}-invariant, since $\mathrm{ad}_{X_\beta} \cdot X_{-\beta} \subset \mathfrak{g}_0$, and for $\alpha \geq 0$ $\mathrm{ad}_{X_\beta} \cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha+\beta}$ and $\alpha + \beta \geq 0$.

(2) V is $\mathrm{ad}_{X_{-\beta}}$-invariant, since $\mathrm{ad}_{X_{-\beta}} \cdot X_{-\beta} = 0$, $\mathrm{ad}_{X_{-\beta}} \cdot \mathfrak{t}_\mathbb{C} \subset \mathbb{C}X_{-\beta}$ and for any $\alpha > 0$ we have $\alpha \geq \beta$ so $\mathrm{ad}_{X_{-\beta}} \cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha-\beta}$ with $\alpha - \beta \geq 0$.

Since the adjoint representation is a Lie algebra representation (Corollary 108), $\mathrm{ad}_{H_\beta} = \left[\mathrm{ad}_{X_\beta}, \mathrm{ad}_{X_{-\beta}}\right]$ so V is also stable by ad_{H_α}. This being a commutator in $\mathrm{End}_\mathbb{C}(V)$ we get that $\mathrm{Tr}_\mathbb{C}\left(\mathrm{ad}_{H_\beta} \restriction V\right) = 0$. On the other hand, we can compute this trace via the eigenspace decomposition:

$$\mathrm{Tr}_\mathbb{C}\left(\mathrm{ad}_{H_\beta} \restriction V\right) = 2\pi i \beta(H_\beta) + 0 + \sum_{\alpha > 0} \dim_\mathbb{C} \mathfrak{g}_\alpha \cdot 2\pi i \alpha(H_\beta) .$$

Dividing by 2π and rearranging the terms we conclude that

$$\left(\dim_\mathbb{C} \mathfrak{g}_\beta - 1\right) \beta\left(iH_\beta\right) + \sum_{\alpha > \beta} \dim_\mathbb{C} \mathfrak{g}_\alpha \cdot \alpha(iH_\beta) = 0 .$$

Now $iH_\beta \in \mathfrak{t}$ is a non-zero multiple of H making the numbers $\beta(iH_\beta), \alpha(iH_\beta)$ are either all positive or all negative. Further, the coefficients $\left(\dim_\mathbb{C} \mathfrak{g}_\beta - 1\right), \dim_\mathbb{C} \mathfrak{g}_\alpha$ are all non-negative. It follows that $\dim_\mathbb{C} \mathfrak{g}_\beta = 1$ (hence also that $\dim_\mathbb{C} \mathfrak{g}_{-\beta} = 1$) and that $\dim_\mathbb{C} \mathfrak{g}_\alpha = 0$ if $\alpha > \beta$, or in other words that $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_{-\beta} \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{g}_\beta$ is three dimensional. □

REMARK 143. It was shown along the same lines in the Lie Algebras course that a complex semisimple Lie algebra of rank 1 is ${}_2\mathbb{C}$.

3.5.4. The algebraic Weyl group. Continuing with our general group G and maximal torus T, let $\alpha \in \Phi$ and let $\mathfrak{u}_\alpha = \ker(\alpha)$, a codimension-1 subspace of $\mathfrak{t}$, $G_\alpha = Z_G(\mathfrak{u}_\alpha)$.

LEMMA 144. $\mathfrak{u}_\alpha$ *is the Lie algebra of the kernel of the exponential root* χ_α. *In particular,* $\exp(\mathfrak{u}_\alpha)$ *is a closed subgroup of* T *of codimension* 1.

REMARK 145. That kernel need not be connected (for example, the kernel of the root of $\mathrm{SU}(2)$ consists of the disconnected centre). We will later see that this kernel has at most two connected components.

PROPOSITION 146. G_α *is a connected subgroup of semisimple rank* 1. *Moreover:*

(1) $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha} = 1$ *and* $\pm\alpha$ *are the only roots proportional to* α.
(2) $W(G_\alpha : T) \simeq C_2$.
(3) *Let* $s_\alpha \in W(G_\alpha : T) \subset W(G : T)$ *be the non-trivial element. Then* $s_\alpha \in \mathrm{GL}(\mathfrak{t})$ *is a reflection in the hyperplane* u_α.

PROOF. G_α centralizes the Lie algebra of a torus, so by Lemma 126 it is connected. Since $T \supset \exp(\mathfrak{u}_\alpha)$ is commutative, we see that $T \subset G_\alpha$ so that T is a maximal torus there as well. By construction, $\mathfrak{u}_\alpha \subset Z_{\mathrm{Lie}\, G_\alpha}$ so the semisimple rank is at most 1. It is not zero since G_α is non-commutative: its lie algebra contains both $\mathfrak{t}$ and $\Re(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, and these subspace do not commute.

Set $\bar{G}_\alpha = G_\alpha / \operatorname{Ker}\chi_\alpha$, and let $\bar{T} = T/\operatorname{Ker}(\chi_\alpha)$, a maximal torus there. This is a connected group of rank 1, hence isomorphic to one of $\mathrm{SU}(2), \mathrm{SO}(3)$.

(1) Let β be a root proportional to α. Then $\pm\beta(H) = 0$ for any $H \in \mathfrak{u}_\alpha$ and it follows that $\Re\left(\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}\right) \subset \mathrm{Lie}\, G_\alpha$ and hence that $\mathfrak{g}_\beta \subset \mathrm{Lie}\, \mathbb{C}G_\alpha$. The direct sum over all these subspaces is disjoint from $\mathfrak{u}_\alpha$ so they all inject into $\mathrm{Lie}\, \mathbb{C}G_\alpha / \mathfrak{u}_{\alpha\mathbb{C}}$. Being the complexified Lie algebra of $\bar{G}_\alpha$ it is three-dimensional and it follows that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha} = 1$ and that there are no other roots proportional to α.
(2) If $g \in G_\alpha$ normalizes T then its image in $\bar{G}_\alpha$ normalizes its maximal torus $\bar{T}$. Conversely, if the image of g normalizes $\bar{T}$ then for any $t \in T$ we have $gtg^{-1} \in T\operatorname{Ker}(\chi_\alpha) = T$ so g normalizes T. It follows that the quotient map induces an isomorphism of the Weyl groups $W(G_\alpha; T) \simeq W\left(\bar{G}_\alpha : \bar{T}\right) \simeq C_2$.
(3) Since $\mathfrak{u}_\alpha$ is central in $\mathrm{Lie}\, G_\alpha$ it is fixed by any element of G_α. The non-trivial element of $W\left(\bar{G}_\alpha : \bar{T}\right)$ acts by inversion on $\bar{T}$, so s_α acts by inversion on $\mathfrak{t}/\mathfrak{u}_\alpha$, that is by a reflection in $\mathfrak{u}_\alpha$ on $\mathfrak{t}$.

□

REMARK 147. We call a root *reduced* if it is not a multiple of another root, and we see that here every root is reduced.

Since $N_{G_\alpha}(T) \subset N_G(T)$ we can think of $s_\alpha \in N_{G_\alpha}(T)/T$ as an element of $W = N_G(T)/T$. This element is a reflection on $\mathfrak{t}$ fixing $\mathfrak{u}_\alpha$. Having equipped $\mathfrak{g}$ with an inner product, the Weyl group acts by isometries on $\mathfrak{t}$ so s_α must be the orthogonal reflection in $\mathfrak{u}_\alpha$. We note that W also acts on the dual space $\mathfrak{t}^*$ fixing the dual lattice Λ^* and the roots Φ and that $s_\alpha(\alpha) = -\alpha$.

DEFINITION 148. Call s_α the *root reflection* associated to the root α. We call the subgroup of the Weyl group generated by the root reflections the *algebraic Weyl group*.

COROLLARY 149. *Let* $\mathfrak{z} = Z(\mathfrak{g})$ *be the Lie algebra of the centre of G and let* $V = \{\nu \in \mathfrak{t}^* \mid \nu(\mathfrak{z}) = 0\} = (\mathfrak{t}/\mathfrak{z})^*$. *Then*$(V, \Phi)$ *is a* root system*, in that it has the following properties:*
(1) $\Phi \subset V$ *is a finite set not containing* $\{0\}$.
(2) $\mathrm{Span}_{\mathbb{R}} \Phi = V$.
(3) *For every* $\alpha \in \Phi$*, the reflection* s_α *in the hyperplane perpendicular to* α *preserves* Φ *setwise.*

EXAMPLE 150. Let $G = \mathrm{SU}(3)$. Let $T = \{\operatorname{diag}(e(i\theta_1), e(i\theta_2), e(i\theta_3)) \mid \theta_1 + \theta_2 + \theta_3 = 0\}$. This is a torus (isomorphic to $(S^1)^2$). To see that it is maximal and compute its Weyl group, restrict the standard representation of $\mathrm{SU}(3)$ on $\mathbb{C}^3$ to T. The coordinate axes are exactly the irreducible subrepresentations and they are non-isomorphic (each is one copy of a different character).

It follows that every $w \in N_G(T)$ must permute these subspaces and every $t \in Z_G(T)$ must act on each subspace separately. But these subspaces are irreducible, so each $t \in Z_G(T)$ must be diagonal, and hence an element of t. It follows that $T = Z_G(T)$ so it is a maximal torus, that $N_G(T)$ is the group of signed permutations, and that $W(G:T) = N_G(T)/T \simeq S_3$.

Differentiaing the definition $G = \left\{ g \in \mathrm{SL}_3(\mathbb{C}) \mid g^\dagger g = \mathrm{Id} \right\}$ we see that $\mathfrak{g} = \left\{ X \in_3 \mathbb{C} \mid \mathbb{X}^\dagger + \mathbb{X} = \not{\vdash} \right\}$ that is the set of anti-Hermitian matrices of trace zero. Since every $Y \in_3 \mathbb{C}$ can be uniquely written in the form

$$Y = \frac{Y+Y^\dagger}{2} + \frac{Y-Y^\dagger}{2} = \frac{Y-Y^\dagger}{2} + i\frac{Y+Y^\dagger}{2i} \in \mathfrak{g} \oplus i\mathfrak{g}$$

we see that $\mathfrak{g}_\mathbb{C} \simeq_3 \mathbb{C}$. It is also clear that $\mathfrak{t} = \{ i \operatorname{diag}(\theta_1, \theta_2, \theta_3) \mid \theta_1 + \theta_2 + \theta_3 = 0 \}$.

Now for $i \neq j$ let $E^{ij} \in_3 \mathbb{C} \subset \mathbb{M}_{\not\vDash}(\mathbb{C})$ be the matrix with zeroes everywhere except that $\left(E^{ij}\right)_{ij} = 1$. Then for $H = i \operatorname{diag}(\theta_1, \theta_2, \theta_3)$ we have $\mathrm{ad}_H \cdot E^{ij} = i(\theta_i - \theta_j) E^{ij}$ so the roots of G are the maps $e_{ij}(H) = \theta_i - \theta_j$.

To find the Weyl chamber we note that the Frobenius, or Hilbert–Schmidt norm on $M_3(\mathbb{C})$ is $\mathrm{U}(3)$-invariant. In terms of this norm (and removing the factor of i) an orthonormal basis of $\mathfrak{t}$ is given by $\frac{1}{\sqrt{6}} \operatorname{diag}(1,1,-2), \frac{1}{\sqrt{2}} \operatorname{diag}(1,-1,0)$. Now for

$$H = \frac{x}{\sqrt{6}} \operatorname{diag}(1,1,-2) + \frac{y}{\sqrt{2}} \operatorname{diag}(1,-1,0)$$

we have

$$\begin{aligned} e_{12}(H) &= \sqrt{2} y \\ e_{23}(H) &= \frac{\sqrt{3}}{\sqrt{2}} x - \frac{1}{\sqrt{2}} y \\ e_{13}(H) &= \frac{\sqrt{3}}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y. \end{aligned}$$

In the coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ we therefore have:

$$\mathfrak{u}_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\perp, \mathfrak{u}_{23} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}^\perp, \mathfrak{u}_{13} = \begin{pmatrix} \sqrt{3}/2 \\ +1/2 \end{pmatrix}^\perp.$$

These three lines are the lines at slopes $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ through the origin, dividing $\mathbb{R}^2$ into six identical sectors. We call these sectors *Weyl chambers*, the lines *walls*, and note that S_3 (which has order 6) acts on the six chambers simply transitively.

EXERCISE 151. Do the same for $\mathrm{SU}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(n)$.

3.6. Weyl chambers

3.6.1. Weyl chambers. The complement of hyperplane $\mathfrak{u}_\alpha$ consists of two half-spaces: the sets $\{H \in \mathfrak{t} \mid \alpha(H) > 0\}$ and $\{H \in \mathfrak{t} \mid \alpha(H) < 0\}$. It follows that the connected components of

$$\mathfrak{t} \setminus \bigcup_{\alpha \in \Phi} \mathfrak{u}_\alpha$$

are interections of half-spaces, hence convex cones.

DEFINITION 152. These connected components are called the (open) *Weyl chambers* in $\mathfrak{t}$. We call $\mathfrak{u}_\alpha$ a *wall* of the chamber C if $\dim\left(\mathfrak{u}_\alpha \cap \bar{C}\right) = \mathrm{rk}\, G - 1$. More generally, a (codimension-k-) *facet* of the Weyl chamber C is any non-empty set of the form $F = \left(\mathfrak{u}_{\alpha_1} \cap \cdots \mathfrak{u}_{\alpha_k} \cap \bar{C}\right)^\circ$ where the interior is taken as a subset of the vector space $\mathfrak{u}_{\alpha_1} \cap \cdots \mathfrak{u}_{\alpha_k}$. We note that the closure $\bar{C}$ is the disjoint union of the facets of C (where C itself is the unique facet of codimension zero).

Equivalently, for each $f \colon \Phi \to \{\pm, 0\}$ we have a facet $F_f = \{H \in \mathfrak{t} \mid \forall \alpha \in \Phi : \mathrm{sgn}\,(\alpha(H)) = f(\alpha)\}$ (excluding those f for which F_f is empty) with open chambers corresponding to functions valued in $\{\pm\}$.

REMARK 153. Note that we are studying the Weyl chambers in $\mathfrak{t}$, rather than the Weyl chambers in $\mathfrak{t}^*$ where the root system lies.

Given a chamber C, let Δ be the set of roots α such that $\mathfrak{u}_\alpha$ is a wall of C and such that α is positive on C (note that $\mathfrak{u}_\alpha = \mathfrak{u}_{-\alpha}$ and that exactly one of $\alpha, -\alpha$ is positive on C).

LEMMA 154. *The chamber is exactly the set bounded by the walls:* $C = \{H \in \mathfrak{t} \mid \forall \alpha \in \Delta \colon \alpha(H) > 0\}$.

PROOF. See PS7. $\square$

OBSERVATION 155. *The Weyl group acts on G by automorphisms while fixing T. It therefore permutes the roots, hence their kernels, and hence the Weyl chambers.*

PROPOSITION 156. *The group* $W' = \left\langle \{s_\alpha\}_{\alpha \in \Delta} \right\rangle$ *acts transitively on the set of Weyl chambers.*

PROOF. Fix $x \in C$; let C' be any other chamber and let $y \in C'$. Note that (being equivalence classes for an equivalence relation) if two chambers intersect they are equal, to it suffices to show that $wy \in C$ for some $w \in W'$. For this choose w such that $\|wy - x\|$ is minimal. If $wy \notin C$ then by Lemma 154 above, there is a wall $\mathfrak{u}_\alpha$ such that x, wy are on opposite sides of α. Decomposing x, wy into their components along and perpendicular to $\mathfrak{u}_\alpha$ it is then clear that

$$\|s_\alpha(wy) - x\| < \|wy - x\| \,,$$

which is a contradiction since $s_\alpha w \in W'$. $\square$

PROPOSITION 157. *The group W acts simply transitively on the chambers.*

PROOF. We already know the action is transitive. Suppose $w \in N_G(T)$ stabilizes the chamber C. Since W is finite, w has finite order as an automorphism of T so averaging over a w-orbit shows that w fixes some $x \in C$ (recall that C is convex). Think of x as an element $H \in \mathfrak{t}$, we have that $\mathrm{Ad}_w \cdot H = H$, that is $w \in Z_G(H)$.

On the other hand, since $H \in C$, $\alpha(H) \neq 0$ for all $\alpha \in \Phi$. It follows that H acts non-trivially in every root space so $Z_{\mathfrak{g}_\mathbb{C}}(H) = \mathfrak{t}_\mathbb{C}$ and hence $Z_\mathfrak{g}(H) = \mathfrak{t}$. Now $Z_G(H)$ is connected (this is true for all $H \in \mathfrak{g}$); its lie Algebra being $Z_\mathfrak{g}(H)$ we conclude that $Z_G(H) = T$ and hence that $w \in T$. It follows that the image of w in $W = N_G(T)/T$ is trivial. $\square$

THEOREM 158. $W' = W$*, that is the algebraic and analytic Weyl groups coincide.*

PROOF. Let $w \in W$. By the transitivity of W' there is $w' \in W'$ such that $w \cdot C = w' \cdot C$. By the simplicity of the action we conclude $w = w' \in W'$. $\square$

COROLLARY 159. *For any* $H \in \mathfrak{t}$, $\mathrm{Stab}_W(H) = \langle s_\alpha \mid \alpha(H) = 0 \rangle$. *In other words, the stabilizer of H is generated by reflections in the hyperplanes containing it.*

PROOF. Problem set. $\square$

3.6.2. Geometry of the roots. The linear map $s_\alpha - \mathrm{Id}_{\mathfrak{t}}$ is non-zero but vanishes on $\mathfrak{u}_\alpha$. It therefore has rank 1, and factors through α. We conclude that there is a unique $\check{\alpha} \in \mathfrak{t}$ such that

$$s_\alpha(x) = x - \alpha(x)\check{\alpha}\,.$$

The dual action on $\mathfrak{t}^*$ is then

$$s_\alpha(\nu) = \nu - \nu(\check{\alpha})\alpha$$

and since $s_\alpha(\alpha) = -\alpha$ we have $\alpha(\check{\alpha}) = 2$.

DEFINITION 160. Call $\check{\alpha}$ the *coroot* associated to α and write $\check{\Phi}$ for the set of coroots.

REMARK 161. If $\alpha+\beta$ is a root it need not be the case that $\check{\alpha+\beta} = \check{\alpha}+\check{\beta}$. In particular, a root system and its dual need not be isomorphic.

EXERCISE 162. $\left(\mathfrak{t}/\mathfrak{z}, \check{\Phi}\right)$ is a root system, the *dual root system*.

LEMMA 163. $\mathrm{Ker}\,\chi_\alpha$ *is central in* G_α.

PROOF. Let $z \in \mathrm{Ker}\,\chi_\alpha$. Then $\mathrm{Ad}(z)$ acts trivially on $\mathrm{Lie}\,G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ (see Proposition 146), thus on the connected group G_α. □

PROPOSITION 164. (1) *Coroots are integral, that is* $\check{\alpha} \in \Lambda = \mathrm{Ker}(\exp\restriction_{\mathfrak{t}})$.
(2) *(not for class) Let* $\iota\colon T \to G$ *be the inclusion map,* $\iota_*\colon \pi_1(T) \to \pi_1(G)$ *the induced map on fundamental groups. Identifying* $\pi_1(T) = \Lambda$ *we have* $\iota_*(\check{\alpha}) = 1$.

PROOF. Since $T \subset G_\alpha$ and the inclusion map into G factors through G_α as well, we may assume $G = G_\alpha$.

For the first claim since $\alpha(\check{\alpha}) = 2$, the element $\frac{1}{2}\check{\alpha}$ has

$$\chi_\alpha\left(\exp(\frac{1}{2}\check{\alpha})\right) = \exp\left(2\pi i\alpha(\frac{1}{2}\check{\alpha})\right) = \exp\left(2\pi i\right) = 1\,.$$

By the Lemma this means $\exp\left(\frac{1}{2}\check{\alpha}\right)$ is central in G and hence is fixed by s_α. On the other hand, $s_\alpha(\check{\alpha}) = -\check{\alpha}$ so $\mathrm{Ad}(s_\alpha)\exp\left(\frac{1}{2}\check{\alpha}\right) = \exp\left(-\frac{1}{2}\check{\alpha}\right)$. It follows that

$$\exp\left(\frac{1}{2}\check{\alpha}\right) = \exp\left(-\frac{1}{2}\check{\alpha}\right)\,,$$

that is $\exp\left(\check{\alpha}\right) = 1$ and $\check{\alpha} \in \Lambda$.

The class of $\check{\alpha}$ in $\pi_1(T)$ is represented by the path $t \mapsto \exp(t\check{\alpha})$ ($t \in [0,1]$) and we would like to show it is nulhomotopic in G_α, or requivalently that the restriction to $\left[\frac{1}{2},1\right]$ is homotopic to the reverse of the restriction to $\left[0,\frac{1}{2}\right]$. For this note that

$$\exp\left((1-t)\check{\alpha}\right) = \exp(t(-\check{\alpha})) = \mathrm{Ad}\,s_\alpha\left(\exp(t\check{\alpha})\right),$$

and for t ranging from 0 to $\frac{1}{2}$ (so $1-t$ is ranging from 1 to $\frac{1}{2}$) this is homotopic to $t \mapsto \exp(t\check{\alpha})$ since G_α is connected (so we can continuously deform s_α to the identity). □

COROLLARY 165. *For any* $\alpha,\beta \in \Phi$ *we have* $n_{\alpha\beta} \overset{def}{=} \beta\left(\check{\alpha}\right) \in \mathbb{Z}$.

DEFINITION 166. The $n_{\alpha\beta}$ are called the *Cartan numbers* of $\mathfrak{g}$. Note that $s_\alpha(\beta) = \beta - n_{\alpha\beta}\alpha$.

DEFINITION 167. The *coroot lattice* is the subgroup $\Gamma < \Lambda$ generated by the coroots.

FACT 168. $\Lambda/\Gamma \simeq \pi_1(G)$.

COROLLARY 169. *$\tilde{G}$ is compact iff $\pi_1(G)$ is finite iff $\check{\Phi}$ spans $\mathfrak{t}$ iff $\mathfrak{z} = 0$ iff $Z(G)$ is finite. In each of those equivalent cases we say that G is* semisimple.

FACT 170. *G is semisimple iff its lie algebra is the direct sum of nonabelian simple lie algebras, iff G is the almost direct product of nonabelian quasisimple groups.*

Recall that we have equipped $\mathfrak{g}$ with an invariant inner product. This also endows $\mathfrak{t}^*$ with an inner product and then

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

so if we identify $\mathfrak{t}$, $\mathfrak{t}^*$ using this inner product the element $\check{\alpha}$ is identified with $\check{\alpha}^* = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{t}^*$. Now $n_{\alpha\beta} = \beta(\check{\alpha}) = \langle \beta, \check{\alpha}^* \rangle = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$, and it follows from Cauchy–Schwarz that

$$n_{\alpha\beta} n_{\beta\alpha} = 4\frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \leq 4$$

holds, with equality iff α, β are proportional. Since the two Cartan numbers are integers, each is zero iff $\alpha \perp \beta$, and if both are non-zero their product is positive, we see that (up to exchanging α, β) if α, β are not proportional, the pair $\left(n_{\alpha\beta}, n_{\beta\alpha}\right)$ must be one of the seven possibilities:

$$(0,0), \pm(1,1), \pm(1,2), \pm(1,3).$$

In each case the pair $\left(n_{\alpha\beta}, n_{\beta\alpha}\right)$ determines the angle between the roots and (if they are not orthogonal) the ratio of their lengths.

COROLLARY 171. *Let α, β be non-proportional and suppose that that $n_{\alpha\beta} > 0$ (equivalently that $\langle \alpha, \beta \rangle > 0$). Then $\alpha - \beta \in \Phi$.*

PROOF. If $n_{\alpha\beta} > 0$ then either $n_{\beta\alpha} = 1$, at which point $s_\beta(\alpha) = \alpha - \beta \in \Phi$, or $n_{\alpha\beta} = 1$, at which point $s_\beta(\alpha) = \beta - \alpha \in \Phi$ and then $\alpha - \beta \in \Phi$ as well. □

3.6.3. Simple roots. Fix a Weyl chamber C, giving a *notion of positivity*: call $\alpha \in \Phi$ *positive* if it is positive on C, negative otherwise, and write Φ^+, Φ^- for the sets of positive and negative roots. Since roots have constant sign on C it suffices to evaluate them at a fixed $H \in C$.

DEFINITION 172. Call $\alpha \in \Phi^+$ *simple* if it is not a sum of positive roots, and let Δ be the set of simple roots.

REMARK 173. This clearly depends on the choice of C. More on that anon.

LEMMA 174. *Every positive root is a positive sum of simple roots.*

PROOF. Let α be a counterexample with $\alpha(H)$ minimal. Then α is not a simple root, so $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$. But then $\beta(H) + \gamma(H) = \alpha(H)$ shows that $\beta(H), \gamma(H) < \alpha(H)$ so they are sums of positive roots and we have a contradiction. □

PROPOSITION 175. $\Delta \subset \mathfrak{t}^*$ *is linearly independent.*

PROOF. Let $\alpha, \beta \in \Delta$ be distinct. If the angle between them was acute ($\langle \alpha, \beta \rangle > 0$) then by Corollary 171 one of $\alpha - \beta, \beta - \alpha$ would be a positive root and this would make either α or β decomposable. It follows that $\langle \alpha, \beta \rangle \leq 0$ for each pair. they are also all contained in the half-plane $\{v \mid v(H) > 0\}$. We show these two hypotheses suffice to make a set of vectors independent.

Indeed, suppose we have a linear dependence in Δ. We then have disjoint non-empty $A, B \subset \Delta$ and *positive* coefficients $\{a_\alpha\}_{\alpha\in A}, \{b_\beta\}_{\beta\in B}$ such that

$$\sum_{\alpha\in A} a_\alpha \cdot \alpha = \sum_{\beta\in B} b_\beta \cdot \beta \,.$$

Call this vector ν. Then

$$0 \le \langle \nu,\nu\rangle = \sum_{\alpha,\beta} a_\alpha b_\beta \langle \alpha,\beta\rangle \le 0$$

and it follows that $\nu = 0$. We therefore have

$$0 = \nu(H) = \sum_{\alpha\in A} a_\alpha \cdot \alpha(H) > 0\,,$$

a contradiction. □

LEMMA 176. Δ *spans* $(\mathfrak{t}/\mathfrak{z})^*$.

PROOF. Every simple root vanishes on $\mathfrak{z}$, so the same holds for every element of the span. Conversely, the span contains Φ; it follows the common kernel of the span is exactly $\mathfrak{z}$ so the span is exactly $(\mathfrak{g}/\mathfrak{z})^*$. □

COROLLARY 177. $\#\Delta$ *is the semisimple rank.*

COROLLARY 178. $\{\check{\alpha}\}_{\alpha\in\Delta}$ *span* $\mathfrak{t}/\mathfrak{z}$.

PROOF. Identifying $\mathfrak{t}$ with $\mathfrak{t}^*$ via an inner product we have $\check{\alpha}^* = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ so the $\check{\alpha}$ are orthogonal to $\mathfrak{z}$ and are linearly independent being proportional to a basis of their span. □

LEMMA 179. $\{\mathfrak{u}_\alpha\}_{\alpha\in\Delta}$ *are the walls of* C.

PROOF. $\{H \mid \forall\alpha\in\Delta : \alpha(H) > 0\} = \{H \mid \forall\alpha\in\Phi^+ : \alpha(H) > 0\} = C$. Since Δ are independent they are all walls. □

DEFINITION 180. A *system of simple roots* (or *simple system*) is a subset $\Delta\subset\Phi$ such that every root is either the sum of elements of Δ or the negative of such a sum.

COROLLARY 181. *Every system of simple roots is the set of walls of a Weyl chamber, we have a bijection between systems of simple roots, notions of positivity, and Weyl chambers, and the Weyl group acts transitively on simple systems. In particular, every root belongs to a simple system.*

DEFINITION 182. Let $\mathfrak{n}\subset\mathfrak{g}$ be the subalgebra generated by $\{\mathfrak{g}_\alpha\}_{\alpha\in\Delta}$.

LEMMA 183. $\mathfrak{n} = \oplus_{\beta\in P}\mathfrak{g}_\beta$ *for some subset* $\Delta\subset P\subset\Phi^+$*; for each* $\alpha\in\Delta$ *the subalgebra* $\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{n}$ *is* $\mathrm{ad}_{X_{-\alpha}}$*-invariant.*

PROOF. The first claim is immediate (the root spaces are one-dimensional and the commutator of roots spaces corresponding to positive roots is either zero or the whole root space corresponding to a larger positive root). For the second claim clearly $\mathfrak{t}_{\mathbb{C}}$ is $\mathrm{ad}X_{-\alpha}$-invariant; for $\mathfrak{n}$ define $\left\|\sum_{\alpha\in\Delta} n_\alpha\alpha\right\|_1 = \sum_\alpha |n_\alpha|$. If $\beta\in\Delta$ (equivalently, $\beta\in P$ has norm 1) the either $\beta\neq\alpha$ in which case $\mathrm{ad}_{X_{-\alpha}}X_\beta = 0$ since $\beta-\alpha$ is not a root, or $\beta=\alpha$ and then $\mathrm{ad}_{X_{-\alpha}}X_\alpha\in\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$. Continuing by induction on $\|\beta\|_1$ for each $\beta \in P$ of norm at least 2 we have $X_\beta = \left[X_\delta, X_{\gamma'}\right]$ where $\delta \in \Delta$ and $\gamma' \in P$ of norm $\|\beta\|_1 - 1$. We then have $\mathrm{ad}_{X_{-\alpha}} X_\delta \in \mathfrak{t}_\mathbb{C}$ as before, so

$$\begin{aligned}\mathrm{ad}_{X_{-\alpha}}\left[X_\delta, X_{\gamma'}\right] &= \left[\mathrm{ad}_{X_{-\alpha}} X_\delta, X_\gamma\right] + \left[X_\delta, \mathrm{ad}_{X_{-\alpha}} X_\gamma\right] \\ &\in \left[\mathfrak{t}_\mathbb{C}, \mathfrak{g}_\gamma\right] + \left[X_\delta, \mathfrak{t}_\mathbb{C} + \mathfrak{n}\right] \\ &\in \mathfrak{g}_\gamma + \mathfrak{g}_\delta + \mathfrak{n} \subset \mathfrak{n}\end{aligned}$$

since $\mathfrak{n}$is closed under ad_{X_δ} and contains $\mathfrak{g}_\gamma, \mathfrak{g}_\delta$ by assumption. □

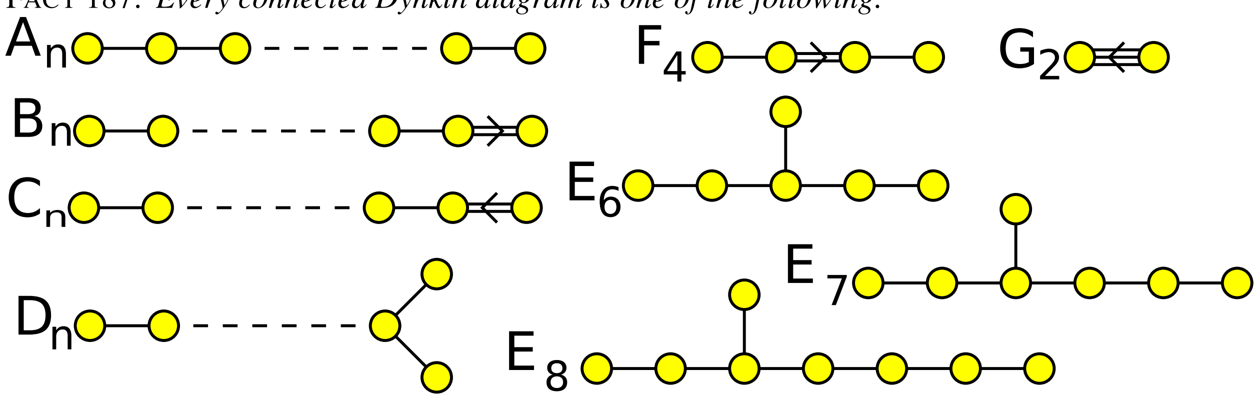
PROPOSITION 184. *We have* $\mathfrak{n} = \bigoplus_{\beta \in \Phi^+} \mathfrak{g}_\beta$.

PROOF. Applying complex conjugation to $\mathfrak{g}$ we get that the subalgebra generated by $\{\mathfrak{g}_{-\alpha}\}_{\alpha \in \Delta}$ is $\bar{\mathfrak{n}} = \oplus_{\beta \in P} \mathfrak{g}_{-\beta}$. The key observation is that $\bar{\mathfrak{n}} \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$. It is clearly $\mathfrak{t}_\mathbb{C}$-invariant (a sum of weight spaces) and by the Lemma it is $\mathrm{ad}_{X_{-\alpha}}$-invariant for each $\alpha \in \Delta$ hence invariant by $\bar{\mathfrak{n}}$. By symmetry it is also $\mathfrak{n}$-invariant. This subalgebra is defined over $\mathbb{R}$ (complex-conjugation-invariant) and its adjoint is compact (has invariant inner product) so the corresponding adjoint group is compact, and its inverse image $H < G$ is closed. Since Δ is still a system of simple roots for H this group has the same Weyl group. It follows that $P \cup (-P)$ is W-invariant and contains Δ, so is all of Φ. Thus $P = \Phi^+$ and $\mathfrak{n}$ is as claimed. □

DEFINITION 185. The *Dynkin diagram* of G (in actuality of $\mathfrak{g}_\mathbb{C}$) is the graph with vertex set Δ. The vertices α, β are connected with $n_{\alpha\beta} n_{\beta\alpha}$ edges which are directed from the short root to the long one (if $\|\alpha\| = \|\beta\|$ we have a single undirected edge).

EXERCISE 186. The lie algebras $\mathfrak{g}_\mathbb{C}/\mathfrak{z}_\mathbb{C}$ and $\mathfrak{g}/\mathfrak{z}$ can be recovered from the Dynkin diagram.

FACT 187. *Every connected Dynkin diagram is one of the following:*[1]



3.6.4. Action of the dual space and the dual Weyl chamber. Recall tht $\Phi \subset \mathfrak{t}^*$ is a *root system*, spanning $(\mathfrak{t}/\mathfrak{z})^*$. Each coroot $\check{\alpha}$ defines a functional on $\mathfrak{t}^*$ by evaluation, and the action of s_α is by reflection in the hyperplane $\{\nu \mid \nu(\check{\alpha}) = 0\}$. The reasoning before shows:

(1) The Weyl group is again W.
(2) Fixing our notion of positivity, we have the *fundamental (closed) Weyl chamber*

$$\mathcal{C} = \{\nu \in \mathfrak{t}^* \mid \forall \alpha \in \Delta : \nu(\check{\alpha}) \geq 0\} = \left\{\nu \in \mathfrak{t}^* \mid \forall \alpha \in \Phi^+ : \langle \nu, \alpha \rangle \geq 0\right\}$$

whose translates by the Weyl group cover $\mathfrak{t}^*$.

[1] Image due to Wikimedia Commons user Tomruen, available at `https://commons.wikimedia.org/wiki/File:Finite_Dynkin_diagrams.svg`

DEFINITION 188. Call $\nu \in \mathfrak{t}^*$ *dominant* if it lies in the fundamental Weyl chamber.

DEFINITION 189. The *fundamental weights* are the basis $\{\omega_i\}_{i=1}^r \subset (\mathfrak{t}/\mathfrak{z})^*$ dual to $\{\check{\alpha}_i\}_{\alpha_i \in \Delta}$. In other words,

$$2\frac{\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}\,.$$

Call a $\nu \in \mathfrak{t}^*$ *algebraically integral* if it is an integral combination of the fundamental weights.

LEMMA 190. *Every integral weight is algebraically integral.*

PROOF. This is Lemma 164 □

REMARK 191. When G is semisimple, the lattice of algebraically integral weights is the dual lattice of the coroot lattice.

3.6.5. The half sum of the positive roots.

LEMMA 192. *Let* $\alpha \in \Delta$, $\beta \in \Phi^+ \setminus \{\alpha\}$. *Then* $s_\alpha(\beta) \in \Phi^+$.

PROOF. Write β as a sum of simple roots. Since $\beta \neq \alpha$ and the root system is reduced, some simple root $\gamma \neq \alpha$ occurs in it. Then $s_\alpha(\beta) = \beta - n_{\alpha\beta}\alpha$ has the same (positive) coefficient of γ, so is a positive root. □

DEFINITION 193. The "half sum of positive roots" is

$$\rho = \frac{1}{2}\sum_{\beta \in \Phi^+} \beta\,.$$

PROPOSITION 194. *Let* $\alpha \in \Delta$, $\beta \in \Phi$, $w \in W$. *Then:*

(1) $s_\alpha \rho = \rho - \alpha$.
(2) $\rho(\check{\alpha}) = 1$.
(3) $w\rho - \rho \in \mathbb{Z}[\Delta]$
(4) $\rho(\check{\beta}) \in \mathbb{Z}$.

PROOF. In order.

(1) $\rho = \frac{1}{2}\alpha + \frac{1}{2}\sum_{\beta \in \Phi^+ \setminus \{\alpha\}} \beta$. Now apply Lemma 192
(2) $s_\alpha(\rho) = \rho - \rho(\check{\alpha})\alpha$.
(3) By induction on the generation of W by the s_α.
(4) We can choose w, α so that $\beta = w\alpha$. Then $\rho(\check{\beta}) = (w\rho)(\check{\alpha}) = \rho(\check{\alpha}) + (w\rho - \rho)(\check{\alpha}) \in \mathbb{Z}$ by (2),(3).

□

Recall the dual positive chamber is defined by $\mathcal{C} = \{\nu \in \mathfrak{t}^* \mid \forall \alpha \in \Delta : \nu(\check{\alpha}) > 0\}$

COROLLARY 195. ρ *lies in the dual positive chamber.*

LEMMA 196. *Let* $\nu \in \Lambda^*$. *Then* $\nu + \rho \in \mathcal{C}$ *iff* $\nu \in \overline{\mathcal{C}}$.

PROOF. We have $(\nu + \rho)(\check{\alpha}) = \nu(\check{\alpha}) + 1 \in \mathbb{Z}$. In particulr, $(\nu + \rho) > 0$ iff $\nu(\check{\alpha}) \geq 0$. □

CHAPTER 4

Representation Theory of Compact Lie Groups

4.1. Setup and preliminary observations

Review.

(1) G a compact connected Lie group with Lie algebra $\mathfrak{g}$
(2) $T \subset G$ a maximal torus with lie algebra $\mathfrak{t}$ and Weyl group $W = N_G(T)/T$.
(3) $\Lambda < \mathfrak{t}$ the integral lattice, $\Lambda^* < \mathfrak{t}^*$ its dual, the weight lattice.
(4) $\Phi = \Phi(G:T) \subset \Lambda^* \subset \mathfrak{t}^*$ the real roots.
(5) $\Delta = \{\alpha_i\}_{i=1}^r \subset \Phi^+$ a system of simple roots corresponding to a Weyl chamber $C \subset \mathfrak{t}$; Φ^+ the set of positive roots.
(6) $\left\{\check{\beta}\right\}_{\beta\in\Phi} \subset \mathfrak{t}$ the coroots, $\{\omega_i\}_{i=1}^r \subset (\mathfrak{t}/\mathfrak{z})^*$ the *fundamental weights*, the basis dual to $\{\check{\alpha}_i\}_{i=1}^r$.
(7) $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$.

We study a finite-dimensional representation $(\pi, V) \in \mathrm{Rep}(G;\mathbb{C})$. Differentiating $\pi\colon G \to \mathrm{GL}(V)$ we obtain a Lie algebra representation $d\pi\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$, which extends naturally to $\mathfrak{g}_{\mathbb{C}}$.

LEMMA 197. *Let $W \subset V$ be a subspace. Then W is G-invariant iff it is $\mathfrak{g}$-invariant.*

PROOF. Let $v \in W$. If W is G-invariant we have $d\pi(X)\cdot v = \frac{d}{dt}\big|_{t=0}\pi(\exp(tX))\,v \in W$. If W is $\mathfrak{g}$-invariant we have

$$\pi(\exp X)\,v = \exp(d\pi(X))\,v = \sum_{k=0}^{\infty}\frac{1}{k!}(d\pi(X))^k\,v \in W\,.$$

Even if G is non-compact this shows that W is fixed by a generating subset of G, hence by all of G. $\square$

4.2. Weights

Restricting π and $d\pi$ to T for each $\mu \in \Lambda^*$ write V_μ for the *weight space*

$$\begin{aligned} V_\mu &= \left\{\underline{v} \in V \mid \forall t \in T : \pi(t)\cdot\underline{v} = \chi_\mu(t)\underline{v}\right\} \\ &= \left\{\underline{v} \in V \mid \forall H \in \mathfrak{t} : H\cdot\underline{v} = 2\pi i\mu(H)\underline{v}\right\} \end{aligned}$$

so that

$$\mathrm{Res}_T^G V = \bigoplus_{\mu\in\Lambda^*} V_\mu\,.$$

LEMMA 198. $\mathfrak{g}_\alpha \cdot V_\mu \subset V_{\mu+\alpha}$.

PROOF. As before for the adjoint representation: Let $X \in \mathfrak{g}_\alpha$, $H \in \mathfrak{t}$, $\underline{v} \in V$. Then

$$\begin{aligned}\pi(H)\left(\pi(X)\underline{v}\right) &= \pi(X)\pi(H)\underline{v} + [\pi(H), \pi(X)]\,\underline{v} \\ &= \pi(X)2\pi i\mu(H)\underline{v} + \pi\left([H,X]\right)\underline{v} \\ &= 2\pi i\mu(H)\pi(X)\underline{v} + \pi\left(2\pi i\alpha(H)X\right)\underline{v} \\ &= 2\pi i\left(\mu(H) + \alpha(H)\right)\left(\pi(X)\underline{v}\right) \\ &= 2\pi i\left(\mu+\alpha)(H)\right)\left(\pi(X)\underline{v}\right).\end{aligned}$$

□

4.2.1. Example: $\mathfrak{su}(2)$ (hence $\mathrm{SU}(2)$, $\mathrm{SO}(3)$ and $\mathrm{SL}_2(\mathbb{C})$). Let $G = \mathrm{SU}(2)$, $\mathfrak{g} = \mathfrak{su}(2)$. Then $\mathfrak{g}_\mathbb{C} =_2 \mathbb{C}$ (see Section 3.4.3 and Example 150). In the standard embedding $\mathrm{SU}(2) \subset \mathrm{SL}_2\mathbb{C}$, we have $\mathfrak{su}(2)$ consist of the anti-hermitian elements of $\mathfrak{sl}_2\mathbb{C} = \{X \in M_2(\mathbb{C}) \mid \mathrm{tr}\,X = 0\}$. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note the usual commutation relations

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h \tag{4.2.1}$$

so our unique root α has $\alpha(h) = 2$ (and h is the coroot!).

THEOREM 199. *The Lie algebra $\mathfrak{sl}_2\mathbb{C} = \mathfrak{su}(2)_\mathbb{C}$ has a unique irreducible representation of each dimension $n \geq 1$.*

PROOF (UNIQUENESS). Let (π, V) be a complex irrep of dimension $n = 2\ell + 1$ ($\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$). Then $\pi(h) \in \mathrm{End}_\mathbb{C}(V)$ has at least one eigenvalue; among the eigenvalues of $\pi(h)$ let λ has maximal real part ("highest weight") and let $\underline{v} \in V$ be an eigenvector:

$$\pi(h)\underline{v} = \lambda\underline{v}.$$

Since $\underline{v} \in V_\lambda$ we have $\pi(e)\underline{v} \in V_{\lambda+2}$. But by the choice of λ, $\lambda+2$ is not an eigenvalue of $\pi(h)$, so that $\pi(e)\underline{v} = 0$. Write $\underline{v}_\ell = \underline{v}$ and for $m = \ell - j$, $j \in \mathbb{Z}_{\geq 0}$ set

$$\underline{v}_m = \pi(f)^j \underline{v}_\ell.$$

By construction we have $\pi(h)\underline{v}_m = (\lambda + 2(\ell - m))\,\underline{v}_m$ and in particular they are linearly independent as long as they are non-zero. It follows that there is a smallest ℓ' such that $\pi(f)\underline{v}_{-\ell'} = \underline{0}$. By construction we then have:

$$\begin{aligned}\pi(h)\underline{v}_m &= (2m + \lambda - 2\ell)\,\underline{v}_m, \\ \pi(f)\underline{v}_m &= \underline{v}_{m-1}.\end{aligned}$$

We now show by induction that

$$\pi(e)\underline{v}_m = (\ell - m)(\lambda - \ell + 1 + m)\underline{v}_{m+1}. \tag{4.2.2}$$

This holds for $m = \ell$ (since $\pi(e)\underline{v}_\ell = \underline{0}$). Suppose this holds for $\underline{v}_m$. Then

$$\begin{aligned}\pi(e)\underline{v}_{m-1} = \pi(e)\pi(f)\underline{v}_m &= \pi(f)\pi(e)\underline{v}_m + [\pi(e), \pi(f)]\,\underline{v}_m \\ &= \pi(f)(\ell - m)(\lambda - \ell + 1 + m)\underline{v}_{m+1} + \pi(h)\underline{v}_m \\ &= [(\ell - m)(\lambda - \ell + 1 + m) + 2m + \lambda - 2\ell]\,\underline{v}_m \\ &= (\ell - m + 1)(\lambda - \ell + m)\underline{v}_m.\end{aligned}$$

It follows that $\operatorname{Span}\{\underline{v}_m\}_{m=-\ell'}^{m=\ell}$ is an $\mathfrak{sl}_2\mathbb{C}$-invariant subspace of V, of dimension $\ell+\ell'+1\geq 1$ (note that ℓ' may be negative). By irreducibility this is all of V – which means that the dimension is also $2\ell+1$ and $\ell'=\ell$. Next, $\underline{v}_{-\ell-1}=\underline{0}$ but $\underline{v}_{-\ell}\neq 0$ so

$$(2\ell+1)(\lambda-2\ell)=0$$

and $\lambda=2\ell'$. In the basis $\{\underline{v}_m\}_{m=-\ell}^{m=\ell}$ we then have

$$\begin{cases}\pi(h)\underline{v}_m & =2m\underline{v}_m\\ \pi(f)\underline{v}_m & =\underline{v}_{m-1}\\ \pi(e)\underline{v}_m & =(\ell-m)(\ell+1+m)\underline{v}_{m+1}\end{cases}\tag{4.2.3}$$

(with the provisos that $\pi(e)\underline{v}_\ell=\pi(f)\underline{v}_{-\ell}=\underline{0}$). □

PROOF (EXISTENCE, VERSION 1). Verify by hand that the maps $\pi(e),\pi(f),\pi(h)$ defined in Equation (4.2.3) satisfy the commutation relations of Equation (4.2.3). □

PROOF (EXISTENCE, VERSION 2). Let $\mathbb{C}[x,y]$ be the graded ring of polynomials in two variables. The group $\mathrm{SL}_2(\mathbb{C})$ acts by change of variables: $(g\cdot P)\begin{pmatrix}x\\y\end{pmatrix}=P\left(g^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)$. Let $V_{d+1}\subset\mathbb{C}[x,y]$ be the subspaces of polynomials which are homogenous of degree d, an $\mathrm{SL}_2(\mathbb{C})$-invariant subspace. The monomials $\left\{x^iy^j\mid i+j=d\right\}$ which span V_{d+1} are the joint eigenvectors for the diagonal torus and have distinct eigenvalues, so a subrepresentation must be spanned by a subset of these monomials. Now the action of unipotent matrices $\begin{pmatrix}1&1\\&1\end{pmatrix}$ (resp. $\begin{pmatrix}1&\\1&1\end{pmatrix}$) show that a subspace containing the monomial x^iy^j must also contain all monominals x^ay^b with $a<i$ (resp. $a>i$) so the representations are irreducible. □

COROLLARY 200. *The second proof gives more: every irreducible representation of* $\mathfrak{sl}_2\mathbb{C}$ *integrates to a representation of* $\mathrm{SL}_2(\mathbb{C})$*, in particular of* $\mathrm{SU}(2)$ *(the last claim already follows from the simple connecteness of* $\mathrm{SU}(2)$*).*

PROPOSITION 201. *Every finite-dimensional representation of* $\mathfrak{sl}_2\mathbb{C}$ *is completely reducible.*

PROOF 1 (LIE GROUP METHOD; "WEYL UNITARY TRICK"). Let $\pi\colon\mathfrak{sl}_2\mathbb{C}\to\operatorname{End}_{\mathbb{C}}V$ be a representation. Restricting π to $\mathfrak{su}(2)\subset\mathfrak{sl}_2\mathbb{C}$ and using the simple connectedness of $\mathrm{SU}(2)$ we see that π integrates to a representaiton $\pi\colon\mathrm{SU}(2)\to\mathrm{GL}(V)$. We know that this representation is completely reducible (Theorem (23)). Each $\mathrm{SU}(2)$-invariant subspace is $\mathfrak{su}(2)$-invariant, hence $\mathfrak{su}(2)_{\mathbb{C}}=\mathfrak{sl}_2\mathbb{C}$-invariant, hence $\mathrm{SL}_2(\mathbb{C})$-invariant. □

- It is also possible to prove this with pure Lie-algebra-theoretic method, but we will not discuss this in lecture.

PROOF 2 (LIE ALGEBRA METHOD). Let $\pi(\omega)\stackrel{\text{def}}{=}\frac{1}{2}\pi(h)^2+\pi(e)\pi(f)+\pi(f)\pi(e)\in\operatorname{End}_{\mathbb{C}}(V)$ be the *Casimir element* (more properly the image of the Casimir element of $\mathcal{U}(\mathfrak{sl}_2\mathbb{C})$ by π). It is easy to check that (for any representation!) ω commutes with $\pi(h),\pi(e),\pi(f)$ and hence (1) is constant in any irreducible representation; and (2) its generalized eigenspaces are $\mathfrak{sl}_2\mathbb{C}$-invariant. It's not hard to check that in the $2l+1$-dimensional representation its eigenvalue is $2\ell(\ell+1)$ (let the element act on the highest weight vector), so it follows that those are the only eigenvalues in a finite-dimesional representations and that a decomposition series in each generalized eigenspace has all its factors isomorphic, and it remains to show that the distinct irreducible represnetations can't nontrivially extend themselves.

Assume then that every irreducible subquotient of V has highest weight $\lambda \in \mathbb{Z}_{\geq 0}$ and let V_λ be the corresponding eigenspace with basis $\left\{\underline{v}_{\lambda,i}\right\}_{i=1}^{\dim V_\lambda}$. Defining $\underline{v}_{\lambda,i,j} = \pi(f)^j \underline{v}_{\lambda,i}$ clearly each $\left\{\underline{v}_{\lambda,i,j}\right\}_{0\leq j\leq 2\lambda}$ is an irrep and they are linearly independent: repeately applying $\pi(f)$ pushes a minimal dependence into V_λ. Let U^λ be the sum of these irreps, the subrepresentation generated by V_λ. If V/U^λ is non-trivial it contains an irredicuble representation, necessarily also with highest weight λ, so let $w \in V$ be such a vector. Then $(\pi(h)-\lambda)w \in U^\lambda$. Since $\pi(h)-\lambda$ is invertible on the weight spaces in U^λ other than the highest one, we can modify v by an element of U^λ (i.e. without changing its class in V/U^λ) so that $v = (\pi(h)-\lambda)w \in U^\lambda_\lambda = V_\lambda$. Here v is nonzero since otherwise we'd have $w \in V_\lambda$ contradicting the fact that it is nonzero mod U^λ.

Since w is a highest weight vector in V/U^λ we have $\pi(e)w \in U^\lambda$ and then

$$\begin{aligned}(\pi(h)-(\lambda+2))\,\pi(e)w &= \pi(e)\,(\pi(h)-(\lambda+2))\,w + [\pi(h),\pi(e)]\,w \\ &= \pi(e)v - 2\pi(e)w + 2\pi(e)w = 0\end{aligned}$$

since v is a highest-weight vector in U^λ. But $\lambda+2$ is not an eigenvalue of $\pi(h)$, so $\pi(e)w = 0$ exactly.

CLAIM 202. For all $j \geq 0$ we have

$$\begin{aligned}\pi(h)\pi(f)^j - \pi(f)^j\pi(h) &= -2j\pi(f)^j \\ \pi(e)\pi(f)^j - \pi(f)^j\pi(e) &= j\pi(f)^{j-1}\,(\pi(h)-(j-1))\end{aligned}$$

PROOF OF CLAIM. For $j=0$ both are immediate. Assuming the claims for j we have

$$\begin{aligned}\pi(h)\pi(f)^{j+1} - \pi(f)^{j+1}\pi(h) &= \left(\pi(h)\pi(f)^j - \pi(f)^j\pi(h)\right)\pi(f) + \pi(f)^j\,(\pi(h)\pi(f)-\pi(f)\pi(h)) \\ &= -2j\pi(f)^j\pi(f) + \pi(f)^j(-2\pi(f)) \\ &= -2(j+1)\pi(f)^{j+1}\,.\end{aligned}$$

$$\begin{aligned}\pi(e)\pi(f)^{j+1} - \pi(f)^{j+1}\pi(e) &= \left(\pi(e)\pi(f)^j - \pi(f)^j\pi(e)\right)\pi(f) + \pi(f)^j\,(\pi(e)\pi(f)-\pi(f)\pi(e)) \\ &= j\pi(f)^{j-1}\pi(h)\pi(f) - j(j-1)\pi(f)^j + \pi(f)^j\pi(h) \\ &= (j+1)\pi(f)^j\pi(h) + j\pi(f)^{j-1}\,[\pi(h),\pi(f)] - j(j-1)\pi(f)^j \\ &= (j+1)\pi(f)^j\pi(h) - \pi(f)^j(j^2-j+2j) \\ &= (j+1)\pi(f)^j\,(\pi(h)-(j+1-1))\end{aligned}$$

□

Now for each j, $\pi(f)^j w$ is a vector of weight $\lambda-2j$ in V/U^λ; in fact it is in the generalized $(\lambda-2j)$-eigenspace of $\pi(h)$

$$\begin{aligned}(\pi(h)-(\lambda-2j))\,\pi(f)^j w &= \pi(f)^j\,(\pi(h)-(\lambda-2j))\,w + \left[\pi(h),\pi(f)^j\right]w \\ &= \pi(f)^j v + 2j\pi(f)^j w - 2j\pi(f)^j w \\ &= \pi(f)^j v\,,\end{aligned}$$

so

$$(\pi(h)-(\lambda-2j))^2\,\pi(f)^j w = (\pi(h)-(\lambda-2j))\,\pi(f)^j v = 0\,.$$

Similarly to the case of $\pi(e)w$, $-\lambda-2$ is not an eigenvalue of $\pi(h)$, so we must have $\pi(f)^{\lambda+1}w = 0$. But then both $\pi(e)\pi(f)^{\lambda+1}w = 0$ and $\pi(f)^{\lambda+1}\pi(e)w = 0$, so by the second part of the claim

$$0 = (\lambda+1)\pi(f)^{\lambda}\,(\pi(h)-\lambda)\,w = (\lambda+1)\,\pi(f)^{\lambda}\,v\,.$$

But $\pi(f)^{\lambda}v$ is non-zero (it is the lowest weight vector in the irrep generated by v) and $\lambda+1\geq 1\neq 0$, a contradiction. $\square$

COROLLARY 203. *Every finite-dimensional representation of ${}_2\mathbb{C}$ is a sum of weight spaces, that is $\pi(h)$ is diagonable there.*

REMARK 204. The theory of Jordan decomposition in algebraic groups shows that the image of a semisimple element by an algebraic representation is always semisimple. We have given a concrete proof in this particular case (that every irrep is algebraic follows from the explicit construction above).

REMARK 205. The same arguments would apply to a group of *semisimple* rank 1.

4.3. Theory of the heighest weight

4.3.1. Algebraic preliminary: the universal enveloping algebra. Recall that for a discrete group G and a field F, its *group ring* is the non-commutative algebra $F[G]$ defined as the F-vector space with basis G and with structure constants coming from group multiplication: $g\cdot h = gh$. It is then easy to check that every representation of G on an F-vectorspace endows it with the structure of an $F[G]$-module, and conversely every $F[G]$ module restricts to a representation of G. Under this equivalence of categories note, for example, that if $(\pi,V)\in \mathrm{Rep}_F(G)$ then the subrepresentation generated by $\underline{v}\in V$ is exactly $F[G]\cdot\underline{v}$.

To deal with representations of a general Lie algebra $\mathfrak{g}$ we need the Lie theory analogue of this construction, called the *universal enveloping algebra.*

DEFINITION 206. Let V be an F-vectorspace. The *tensor algebra* is the vector space

$$T(V) \stackrel{\text{def}}{=} \bigoplus_{n\geq 0} V^{\otimes n}$$

equipped with the graded algebra structure coming from the maps $V^{\otimes k}\otimes V^{\otimes \ell}\to V^{\otimes (k+\ell)}$. Write $\iota\colon V\to T(V)$ for the isomorphism $V\to V^{\otimes 1}$.

LEMMA 207. *Let $B=\{\underline{v}_i\}_{i\in I}\subset V$ be a basis. For $\sigma\colon [n]\to I$ let $v_\sigma = \underline{v}_{\sigma(0)}\otimes\cdots\otimes\underline{v}_{\sigma(n-1)}$. then for each n $\{\underline{v}_\sigma\}_{\sigma\in I^n}\subset V^{\otimes n}$ is an F-basis, so $\bigsqcup_{n=0}^{\infty}\{\underline{v}_\sigma\}_{\sigma\in I^n}$ is an F-basis of $T(V)$.*

PROPOSITION 208. *The tensor algebra is a unital associative F-algebra generated by V (in fact by any basis of V). It fulfills the following universal property: for every associative F-algebra A, every F-linear map $f\colon V\to A$ extends to a unique F-algebra homomorphism $\tilde{f}\colon T(V)\to A$ so that $\tilde{f}\circ\iota = f$.*

PROOF. For each n define $f_n\colon V^n\to A$ by $f_n(v_1,\ldots,v_n) = f(v_1)f(v_2)\cdots f(v_n)$. This is clearly n-linear so extends uniquely to a linear map $\tilde{f}_n\colon V^{\otimes n}\to A$. Let $\tilde{f} = \oplus_n\tilde{f}_n$. That $\tilde{f}$ is an algebra homomorphism follows from the universal property of the isomorphism $V^{\otimes k}\otimes V^{\otimes \ell}\to V^{\otimes(k+\ell)}$, and it is clear that $\tilde{f}\circ i = \tilde{f}_1\circ i = f$. Uniqueness follows from the fact that V generates $T(V)$. $\square$

DEFINITION 209. Suppose now that $\mathfrak{g}$ is a Lie algebra, and $J \subset T(\mathfrak{g})$ be the two-sided ideal generated by the relations $X \otimes Y - Y \otimes X = [X,Y] \in \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \subset T(\mathfrak{g})$. The *uinversal enveloping algebra* is the associative F-algebra

$$\mathcal{U}(\mathfrak{g}) \stackrel{\text{def}}{=} T(\mathfrak{g})/J\,.$$

Write $\iota\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ for the composition of the inclusion $\mathfrak{g} \subset T(\mathfrak{g})$ and the quotient map.

LEMMA 210. *$\mathcal{U}(\mathfrak{g})$ is unital, in particular non-trivial.*

PROOF. J is contained in the maximal ideal $\bigoplus_{n\geq 1} V^{\otimes n}$. □

PROPOSITION 211. *For every associative F-algebra A and any Lie algebra homomorphism $f\colon \mathfrak{g} \to A$ there exists a unique $\tilde{f}\colon \mathcal{U}(\mathfrak{g}) \to A$ so that $\tilde{f} \circ \iota = f$.*

PROOF. Follows from the universal property of $T(\mathfrak{g})$ and the fact that the kernel of the extension of any such f to $T(V)$ contain J.

Now let $\{X_i\}_{i\in I} \subset \mathfrak{g}$ be an *ordered* basis (i.e. equip I with a linear order). For $\sigma\colon [n] \to I$ write X_σ the image of the corresponding basis element of $T(V)$, the product $X_{\sigma(0)}X_{\sigma(1)}\cdots X_{\sigma(n-1)}$. If $\rho\colon [m] \to I$ is another map write $\sigma \star \rho$ for the *concatenation*

$$(\sigma \star \rho)(k) = \begin{cases} \sigma(k) & k < n \\ \rho(k-n) & n \leq k < n+m\,. \end{cases}$$

Then $X_\sigma X_\rho = X_{\sigma\star\rho}$. □

PROPOSITION 212. *B_{PBW} spans $\mathcal{U}(\mathfrak{g})$.*

PROOF. For $\sigma\colon [n] \to I$ define $W(\sigma) = (n, \#\{k < l \mid \sigma(k) > \sigma(l)\})$ equipped with the lexicographic order. We show by induction on $W(\sigma)$ that for all n and all $\sigma\colon [n] \to I$, $X_\sigma \in \operatorname{Span}_F(B_{\mathrm{PBW}})$. First, $W(\sigma) = (0,0)$ only for the empty function and that $X_\sigma \in B_{\mathrm{PBW}}$. For other $\sigma\colon [n] \to I$ we may suppose that σ is not nondecreasing. In that case there is $0 \leq k < n-1$ so that $\sigma(k+1) < \sigma(k)$ and we have $\sigma = \tau_1 \star \rho \star \tau_2$ where τ_1 is the prefix of σ of length k, $\rho = \begin{pmatrix} 0 & 1 \\ \sigma(k) & \sigma(k+1) \end{pmatrix}$ and τ_2 is the suffix of length $n-k-2$. Write also $\bar{\rho} = \begin{pmatrix} 0 & 1 \\ \sigma(k+1) & \sigma(k) \end{pmatrix}$ and observe that $W(\tau_1 * \bar{\rho} \star \tau_2) = W(\sigma) - (0,1)$: only one inversion is changed between $\tau_1 * \rho \star \tau_2, \tau_1 * \bar{\rho} \star \tau_2$. Then

$$\begin{aligned} X_\sigma &= X_{\tau_1} X_{\sigma(k)} X_{\sigma(k+1)} X_{\tau_2} \\ &= X_{\tau_1}\left(X_{\sigma(k)}X_{\sigma(k+1)} - X_{\sigma(k+1)}X_{\sigma(k)} + X_{\sigma(k+1)}X_{\sigma(k)}\right) X_{\tau_2} \\ &= X_{\tau_1}\left[X_{\sigma(k)}, X_{\sigma(k+1)}\right] X_{\tau_2} + X_{\tau_1} X_{\sigma(k+1)} X_{\sigma(k)} X_{\tau_2} \\ &= \sum_{i\in I} a_i X_{\tau_1 * i * \tau_2} + X_{\tau_1 * \bar{\rho} \star \tau_2} \end{aligned}$$

where $a_i \in F$ are the coefficients so that $\left[X_{\sigma(k)}, X_{\sigma(k+1)}\right] = \sum_i a_i X_i$ (all but finitely many of the a_i are zero). Now $\tau_1 * i * \tau_2$ have length $n-1$ so $W(\tau_1 * i * \tau_2) < W(\sigma)$ and we've already checked that $W(\tau_1 * \bar{\rho} \star \tau_2) < W(\sigma)$. By induction each of the summands is in the span of B_{PBW}, so the same is true for X_σ. □

COROLLARY 213. *Let* $\mathfrak{g} = \oplus_{i=1}^{r} \mathfrak{g}_i$ *where* $\mathfrak{g}_i$ *are subalgebras. Then* $\mathcal{U}(\mathfrak{g}) = \prod_{i=1}^{r} \mathcal{U}(\mathfrak{g}_i)$.

THEOREM 214 (Poincaré–Birkhoff–Witt). B_{PBW} *is a* basis *of* $\mathcal{U}(\mathfrak{g})$.

COROLLARY 215. *The inclusion* $\iota \colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ *is an embedding.*

REMARK 216. This can also be proved structurally, by constructing a faithful finite-dimensional representation and extending it to $\mathcal{U}(\mathfrak{g})$.

PROOF. (Jacobson Lie Algebras, Thm. V.3) Let $\mathcal{V} \subset T(V)$ be the span of the ordered monomials. We construct a linear map $R \colon T(\mathfrak{g}) \to \mathcal{V}$ such that if σ is nondecreasing, $R(X_\sigma) = X_\sigma$, and such that if $\rho \colon [2] \to I$ then

$$R\left(X_{\tau_1 \star \rho \star \tau_2} - X_{\tau_1 \star \bar{\rho} \star \tau_2}\right) = R\left(X_{\tau_1}\left[X_{\rho(0)}, X_{\rho(1)}\right] X_{\tau_2}\right) .$$

It then follows that $R(J) \subset J$ so R descends to a map $\mathcal{U}(\mathfrak{g}) \to \mathcal{V}$ which is right-inverse to the quotient map $\mathcal{V} \to \mathcal{U}(\mathfrak{g})$. It follows that the quotient map is injective and $B_{\mathrm{PBW}} \subset \mathcal{U}(\mathfrak{g})$ is a basis.

We construct R recursively in the lexicographic order above. Thus let $\sigma \colon [n] \to I$; if σ is nondecreasing there is nothing to do; if not there is an inversion in σ so we decompose $\sigma = \tau_1 \star \rho \star \tau_2$and *define*

$$R\left(X_{\tau_1 \star \rho \star \tau_2}\right) = R\left(X_{\tau_1 \star \bar{\rho} \star \tau_2}\right) + R\left(X_{\tau_1}\left[X_{\rho(0)}, X_{\rho(1)}\right] X_{\tau_2}\right)$$

(recall that $\tau_1 \star \bar{\rho} \star \tau_2$ has one fewer reversal and thus occurs earlier in the order). Since the monomials span $T(\mathfrak{g})$ this will extend to the required linear map as soon as we verify that this uniquely specifies R. Suppose that $\sigma(k+1) > \sigma(k)$ and that $\sigma(\ell+1) > \sigma(\ell)$ for some $k < \ell$. We can then use either inversion to define $R(X_\sigma)$ and need to make sure they agree. Since other indices do not play a role without loss of generality we may assume either $n = 3$ and $k = 0, \ell = 1$ or $n = 4$ and $k = 0$, $\ell = 2$. In the first case we have $a > b > c$; if we define R using the inversion of a, b we get by recursion

$$\begin{aligned}
R(X_a X_b X_c) &= R(X_b X_a X_c) + R([X_a, X_b] X_c) \\
&= R(X_b X_c X_a) + R(X_b [X_a, X_c]) + R([X_a, X_b] X_c) \\
&= R(X_c X_b X_a) + R([X_b, X_c] X_a) + R(X_b [X_a, X_c]) + R([X_a, X_b] X_c) .
\end{aligned}$$

If we use the inversion $b > c$ instead we get

$$\begin{aligned}
R(X_a X_b X_c) &= R(X_a X_c X_b) + R(X_a [X_b, X_c]) \\
&= R(X_c X_a X_b) + R([X_a, X_c] X_b) + R(X_a [X_b, X_b]) \\
&= R(X_c X_b X_a) + R(X_c [X_a, X_b]) + R([X_a, X_c] X_b) + R(X_a [X_b, X_c]) .
\end{aligned}$$

The difference of the two RHS is

$$R([X_b, X_c] X_a - X_a [X_b, X_c]) + R(X_b [X_a, X_c] - [X_a, X_c] X_b) + R([X_a, X_b] X_c - X_c [X_a, X_b])$$

and by recursion this is the image by R of

$$[[X_b, X_c], X_a] + [X_b, [X_a, X_c]] + [[X_a, X_b], X_c] = [[X_b, X_c], X_a] + [[X_c, X_a], X_b] + [[X_a, X_b], X_c]$$

which vanishes by the Jacobi identity.

In the first second case we have $a > b$, $c > d$ and similar arguments work depending on the precise order of a, b, c, d, in each case reducing to the Jacobi identity. $\square$

REMARK 217. In positive characteristic the correct notion is that of a *restricted Lie algebra* with an associated notion of *restricted enveloping algebra* (a quotient of the enveloping algebra) for which the PBW Theorem as stated above does not hold (though formally the theorem above does not use the characteristic of the field).

4.3.2. Uniqueness: highest weight vectors. We study an irreducible finite-dimensional representation (π, V) of $\mathfrak{g}$.

EXERCISE 218. The lexicographic order on $\mathbb{R}^r$ makes it into an ordered group; it also respects rescaling by positive reals.

DEFINITION 219. Say $\nu \in \mathfrak{t}^*$ is *positive* if the first nonzero entry in $(\nu(\check{\alpha}_i))_{i=1}^r$ is positive; say $\nu > \nu'$ if $\nu - \nu'$ is positive (negative). Equivalently, pull the lexicographic order of $\mathbb{R}^r$ to $\mathfrak{t}^*$ by pullback via the map of evaluation at the $\{\check{\alpha}_i\}_{i=1}^r$.

Observe that all the positive roots are positive under this order.

LEMMA 220. *Let ν, ν', ν'' be distinct weights of a finite-dimensional representation V. Then*
(1) *If $\nu > \nu'$ and $\nu' > \nu''$ then $\nu > \nu''$.*
(2) $\nu \restriction_{\mathfrak{z}} = \nu' \restriction_{\mathfrak{z}}$.
(3) *Either $\nu > \nu'$ or $\nu > \nu'$.*

PROOF. Suppose the first non-zero entry of $((\nu - \nu')(\check{\alpha}_i))_{i=1}^r$ is the ith, of $((\nu' - \nu'')(\check{\alpha}_i))_{i=1}^r$ the jth. If $i \neq j$ let k be the smaller of the two. Then the first non-zero entry of

$$\left((\nu - \nu'')(\check{\alpha}_i)\right)_{i=1}^r = \left((\nu - \nu')(\check{\alpha}_i)\right)_{i=1}^r + \left((\nu' - \nu'')(\check{\alpha}_i)\right)_{i=1}^r$$

is the kth and equals the positive entry from the corresponding summand. If $i = j$ then the first non-zero entry is that one, and is the sum of two positive entries.

By Schur's Lemma the centre of $\mathfrak{g}$ must act though a single character. It follows that any different of weights vanishes on $\mathfrak{z}$; if also $(\nu - \nu')(\check{\alpha}_i) = 0$ for all i then $\nu - \nu' = 0$ since the $\{\check{\alpha}_i\}_{i=1}^r$ span $\mathfrak{t}/\mathfrak{z}$ as shown in Corollary 178. $\square$

LEMMA 221. $\pi(\mathfrak{t})$ *are diagonable, that is V is a direct sum of weight spaces.*

PROOF. $\mathfrak{z}$ act by characters by the irreducibility. Restricting the representation to $\mathrm{Lie}\, G_\alpha$ we have that $\pi(\check{\alpha})$ is diagonable by 203 (see Remark 205), and the claim follows that this is a spanning set of a commutative algebra. $\square$

LEMMA 222. *V has a unique highest weight λ, which is dominant.*

PROOF. The existence and uniqueness follow from the previous two Lemmata. Now for $\alpha \in \Delta$ we have that λ is also the highest weight for $\mathrm{Res}^{\mathrm{Lie}\, G}_{\mathrm{Lie}\, G_\alpha} V$, so by the results of Section 4.2.1 (see Remark 205) we have $\lambda(\check{\alpha}) \geq 0$ for all $\alpha \in \Delta$. $\square$

THEOREM 223. *Let λ be the highest weight of V.*
(1) $\dim_{\mathbb{C}} V_\lambda = 1$, *and V_λ generates V.*
(2) $V_\lambda = \{\underline{v} \in V \mid \mathfrak{n} \cdot \underline{v} = \underline{0}\}$.
(3) *Every weight of V has the form $\lambda - \sum_{i=1}^r n_i \alpha_i$ with $n_i \in \mathbb{Z}_{\geq 0}$.*
(4) *For $w \in W$ and $\mu \in \Lambda^*$,* $\dim V_{w\mu} = \dim V_\mu$. *Furthermore all weights satisfy $|\mu| \leq |\lambda|$ with equality only in the Weyl orbit of λ (in fact, all weights lie in the convex hull of this orbit).*
(5) *π is uniquely determined by λ.*

PROOF. For $X \in \mathfrak{g}_\beta$ and $\underline{v} \in V_\lambda$ we have $\pi(X)\underline{v} \in V_{\lambda+\beta}$. If $\beta > 0$ then $\lambda + \beta > \beta$ so $V_{\lambda+\beta} = \{0\}$ and V_λ is annihilated by $\mathfrak{n}$.

Fix a non-zero $\underline{v}_\lambda \in V_\lambda$. Then $\mathcal{U}(\mathfrak{g}_\mathbb{C}) \cdot \underline{v}_\lambda$ is a $\mathfrak{g}$-invariant subspace of V, hence all of V. We have $\mathfrak{g}_\mathbb{C} = \bar{\mathfrak{n}} \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}$ so by PBW $\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}(\bar{\mathfrak{n}}_\mathbb{C})\mathcal{U}(\mathfrak{t}_\mathbb{C})\mathcal{U}(\mathfrak{n}_\mathbb{C})$. Since the non-scalar elements of $\mathcal{U}(\mathfrak{n}_\mathbb{C})$ kill $\underline{v}_\lambda$ and $\mathcal{U}(\mathfrak{t}_\mathbb{C})$ acts on it by scalars, we have $\mathcal{U}(\mathfrak{g}_\mathbb{C}) \cdot \underline{v}_\lambda = \mathcal{U}(\bar{\mathfrak{n}}_\mathbb{C}) \cdot \underline{v}_\lambda$. Now the weights in $\mathcal{U}(\bar{\mathfrak{n}}_\mathbb{C})$ are all integer sums of negative roots (giving (3)). These weights are all negative except for the identity, so $\underline{v}_\lambda$ is the only vector of weight λ which completes the proof of (1).

Next, let $\underline{v} \in V$ be annihilated by all the positive roots, hence by $\mathcal{U}(\mathfrak{n}_\mathbb{C})$ and suppose $\underline{v} \notin V_\lambda$. Decomposing $\underline{v}$ into the weight spaces and substracting the component in V_λ let $\underline{v}_\mu \in V_\mu$ be the highest weight component occuring in $\underline{v}$ other than $\underline{v}_\lambda$. Then $\underline{v}_\mu$ is also annihilated by $\mathcal{U}(\mathfrak{n}_\mathbb{C})$ and rescaled by $\mathcal{U}(\mathfrak{t}_\mathbb{C})$ so generates the subrepresentation $\mathcal{U}(\bar{\mathfrak{n}}_\mathbb{C}) \cdot \underline{v}_\mu$ containing only vectors of weights$\leq \mu$ which is impossible since V is irreducible – so we get (2).

The W-invariance follows immediately from the fact that V is a representation of G, but we can also give a proof using only the Lie algebra by restricting the representation to $\mathrm{Lie}\, G_\alpha$ and noting that the classification of representations of those Lie algebras in Section 4.2.1 shows that (a) $\mathrm{Res}^G_{G_\alpha} V$ is completely reducible; (b) the weights in each irrep of G_α are s_α-invariant. Now let μ be a weight of V, wlog dominant (act by the Weyl group) so that $\langle \mu, \alpha_i \rangle \geq 0$. By (3) we have $\lambda = \mu + \sum_{i=1}^r n_i \alpha_i$ with $n_i \in \mathbb{Z}_{\geq 0}$ so

$$|\lambda|^2 = |\mu|^2 + \left| \sum_{i=1}^r n_i \alpha_i \right|^2 + 2 \sum_{i=1}^r n_i \langle \mu, \alpha_i \rangle \geq |\mu|^2 .$$

Furthermore we have equality only if $\sum_{i=1}^r n_i \alpha_i = 0$ that is if $\lambda = \mu$, and in general if $\mu = w\lambda$ for some $w \in W$.

To show that π is determined by λ we can proceed as in Section 4.2.1, computing the action of the raising operators on the weight vectors, but the following proof (see [**1**]) is simpler. Let V, W be two irreducible representations with highest weight λ and highest weight vectors $\underline{v}_\lambda, \underline{w}_\lambda$. Then $\underline{v}_\lambda + \underline{w}_\lambda \in (V \oplus W)_{\lambda,} = V_\lambda \oplus W_\lambda$ is annihilated by $\mathfrak{n}$ so generates a subrepresentation R of $V \oplus W$ of which it is the unique highest weight vector. Let $\pi_V : V \oplus W \to V$ and $\pi_W : V \oplus W \to W$ be the projections, which are G-invariant. Then $\pi_V(R)$ is a subrepresentation of V containing $\underline{v}_\lambda$, that is all of V. On the other hand $\mathrm{Ker}\, \pi_V \restriction_R = R \cap W$ is a subrepresentation of W not containing $\underline{w}_\lambda$, so $\{\underline{0}\}$. It follows that $\pi_V \restriction_R$ is an isomorphism $R \to V$,and for the same reason $\pi_W \restriction_R : R \to W$ is an isomorphism and V, W are isomorphic. □

4.3.3. Existence: Verma modules.

DEFINITION 224. Call a (possibly infinite-dimensional) $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ module V a *highest weight module* if it is generated (as a module) by a vector $v \in V$ of weight λ such as λ is the highest occuring weight.

The proof of Theorem 223 shows that in that case $V = \mathcal{U}(\bar{\mathfrak{n}}_\mathbb{C})v$, that is V is the sum of finite-dimensional weight spaces with weights $\lambda - \sum_{i=1}^r n_i \alpha_i$, and that V_λ is one-dimensional.

DEFINITION 225. For a weight λ let $\mathbb{C}^\lambda$ be the one-dimensional representation of $\mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}$ (equivalently of its UEA) where $\mathfrak{n}$ acts trivially and $\mathfrak{t}$ acts via λ. The *Verma module* W^λ is the induced module

$$\mathrm{Ind}_{\mathcal{U}(\mathfrak{t}_\mathbb{C})\mathcal{U}(\mathfrak{n}_\mathbb{C})}^{\mathcal{U}(\mathfrak{g}_\mathbb{C})} \mathbb{C}^\lambda \simeq \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{t}_\mathbb{C} \oplus \mathfrak{n})} \mathbb{C}^\lambda \simeq \mathcal{U}(\mathfrak{g}_\mathbb{C})/I_\lambda$$

where I_λ is the left ideal of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ generated by $\mathfrak{h}$ and by the elements $\{H-\lambda(H)\}_{H\in\mathfrak{t}}$.

OBSERVATION 226 (Verma module structure).

(1) This is clearly a highest weight module, universal among all highest weight modules with highest weight λ.
(2) Every proper invariant subspace is the sum of its weight spaces and does not contain the highest weight vector, so the sum of all proper invariant subspaces is a proper invariant subspace – the maximal one.
(3) W^λ has a unique irreducible quotient L^λ, the unique irreducible highest-weight module with highest weight λ.

THEOREM 227. *Suppose λ is algebraically integral and dominant. Then L^λ is finite-dimensional.*

PROOF. Let $v_\lambda \in W^\lambda$ be the highest weight vector, and let $\alpha \in \Delta$. By the calculation of Section 4.2.1 if we normalize $X_\alpha, X_{-\alpha}$ so that $[X_\alpha, X_{-\alpha}] = \check{\alpha}$ we have

$$X_\alpha (X_{-\alpha})^{k+1} v_\lambda = (k+1)(\lambda(\check{\alpha})-k)(X_{-\alpha})^k v_\lambda\,.$$

Since λ is integral and dominant we have $\lambda(\check{\alpha}) \in \mathbb{Z}_{\geq 0}$ and it follows that $(X_{-\alpha})^{\lambda(\check{\alpha})+1} v_\lambda$ is annihilated by X_α. If $\beta \in \Delta\setminus\{\alpha\}$ is another simple root then $X_{-\alpha}$ and X_β commute, so $X_\beta\,(X_{-\alpha})^{\lambda(\check{\alpha})+1} v_\lambda = (X_{-\alpha})^{\lambda(\check{\alpha})+1} X_\beta v_\lambda = 0$ and $(X_{-\alpha})^{\lambda(\check{\alpha})+1} v_\lambda$ is itself a highest-weight vector. It follows that in the quotient L^λ the $\operatorname{Lie} G_\alpha$-submodule generated by v_λ is of dimension $\lambda(\check{\alpha})+1$. Let $M_\alpha \subset L^\lambda$ be the sum of all finite-dimensional $\operatorname{Lie} G_\alpha$-submodules. Then $\mathfrak{g}_\mathbb{C} \otimes_\mathbb{C} M_\alpha$ is also such a module, so the same is true for its image in L^λ, which must then be contained in M_α so $M_\alpha \subset L^\lambda$ is a non-trivial $\mathfrak{g}_\mathbb{C}$-submodule, hence all of L^λ. Now the weights of an irreducible $\operatorname{Lie} G_\alpha$-submodule are s_α-invariant and by the theorem on complete reducibility we conclude that the weights of L^λ are W-invariant ($\dim_\mathbb{C} L^\lambda_{w\mu} = \dim_\mathbb{C} L^\lambda_\mu$). Now each weight space is finite-dimensional (this was already true for W^λ), so it is enough to show that there are finitely many weights, and since the Weyl group is finite that there are finitely many dominant weights. Finally if $\lambda - \sum_{i=1}^r n_i\alpha_i$ is dominant then $0 \leq \langle \rho, \lambda - \sum_{i=1}^r n_i\alpha_i\rangle$ so $0 \leq n_i \leq \frac{\langle\rho,\lambda\rangle}{\langle\rho,\alpha_i\rangle}$ and we are done. $\square$

THEOREM 228. *Suppose further that λ is integral. Then L^λ extends to a representation of G.*

4.4. Characters

4.4.1. Analytical preliminary: the Weyl Integration formula.

DEFINITION 229. Call $H \in \mathfrak{t}$ *singular* if $\beta(H) \in \mathbb{Z}$ for some $\beta \in \Phi$. Let $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{t}$ be the set of *regular* (that is nonsingular) elements.

The set of regular elements is the complement of a the family of hyperplanes $\{\beta(H) = m\}$, so is open, dense, and of full (Lebesgue) measure.

LEMMA-DEFINITION 230. *For $H \in \mathfrak{t}$ either all elements of $H+\Lambda$ are singular or all are. Define T^{reg} to be the set of exponentials of regular elements, again an open dense set of full measure.*

By Corollary 131 the the map $(g,t) \mapsto g^{-1}tg$ is a G-equivariant surjection $q\colon G/T \times T \to G$. By Corollary 159 the map is $\#W$-to-1 on the T^{reg}.

LEMMA 231. *The Jacobian determinant of q at $(e,t)$is*

$$J(t) = \det\left(\mathrm{Id} - \mathrm{Ad}_{t^{-1}} \left|\mathfrak{g}/\mathfrak{t}\right.\right) .$$

PROOF. For $X \in \mathfrak{g}/\mathfrak{t}$, $H \in \mathfrak{t}$ we have

$$\begin{aligned}(\mathrm{Id} - X)\, t\, (\mathrm{Id} + H)\, (\mathrm{Id} + X) - t &= -Xt + tX + tH + \text{error} \\ &= t\left(X + H - \mathrm{Ad}_{t^{-1}} X\right)\end{aligned}$$

. □

- This is non-zero on an open dense set (t off the walls) whose complement has measure zero
- For t off the walls, the inverse image of t is exactly its Weyl orbit.

PROPOSITION 232 (Weyl integration formula). *Let $f \in C(G)$. Then*

$$\#W \int_G f(g)\,\mathrm{d}g = \int_T \left[J(t) \int_G f(gtg^{-1})\,\mathrm{d}g\right] \mathrm{d}t .$$

COROLLARY 233. *Let $f \in C(G)$ be a class function. Then*

$$\int_G f(g) dg = \frac{1}{\#W} \int_T J(t) f(t) dt .$$

4.4.2. Algebraic preliminary: the ring of characters.

LEMMA-DEFINITION 234. *Let $I \subset \mathfrak{t}^*$ be any additive subgroup. Then the group ring $R_I = \mathbb{Z}\left[\{e \circ \mu\}_{\mu \in I}\right]$ is a UFD.*

PROOF. For any finite $A \subset I$ write $R_A = R_{\langle A \rangle}$. The subgroup $\langle A \rangle$ is free of finite rank, hence of the form $\langle B \rangle$ for some $B \subset I$ which are linearly independent over $\mathbb{Q}$. Then $R_A = R_B \simeq \mathbb{Z}\left[\left\{x_i^{\pm}\right\}_{i=1}^{\#B'}\right]$ which is a localization of the polynomial ring $\mathbb{Z}\left[\{x_i\}_{i=1}^{\#B}\right]$. This makes each R_B a UFD, which means the same is true for their direct limitR_I. □

We will apply this to the sets $I = \{\mu \in \mathfrak{t}^* \mid \mu(\Gamma) \subset \mathbb{Z}\} \supset \Lambda^*$ of algebraically and analytically integral weights.

LEMMA 235. *Suppose $f \in R_I$ vanishes on $\beta^{-1}(\mathbb{Z})$ for some $\beta \in I$. Then $e \circ \beta - 1$ divides f.*

COROLLARY 236. *Suppose f vanishes on the singular set. Then $\prod_{\beta > 0} (e \circ \beta - 1)$ divides f.*

PROOF. R_I is a UFD and these are pairwise prime irreducibles. □

We now use the W-invariance of I and Λ^*. By the Coxeter relations, or by taking the determinant of the action on $\mathfrak{t}$, there is a character $\mathrm{sgn}\colon W \to \{\pm 1\}$ mapping each root reflection to -1.

DEFINITION 237. Call $f \in R_I$ *symmetric* if it is W-invariant, *alternating* if $f \circ w = \mathrm{sgn}(w) f$.

OBSERVATION 238. *An alternating function vanishes on the singular set.*

PROOF. If $\beta(H) \in \mathbb{Z}$ then $s_\beta(H) = H - \beta(H)\check{\beta}$ so

$$f(H) = -f\left(s_\beta H\right) = -f\left(H - \beta(H)\check{\beta}\right) = -f(H)$$

since for every $\mu \in I$ we have $\mu\left(H - \beta(H)\check{\beta}\right) = \mu(H) - \beta(H)\mu(\check{\beta}) \in \mu(H) + \mathbb{Z}$. □

EXAMPLE 239. We will use the following two alternating functions:

(1) The *Weyl denominator*

$$\begin{aligned} \delta(H) &= \prod_{\beta\in\Phi^+}\left[e\left(\frac{1}{2}\beta(H)\right) - e\left(-\frac{1}{2}\beta(H)\right)\right] \\ &= e\left(-\rho(H)\right)\prod_{\beta>0}\left[e\left(\langle\beta,H\rangle\right)-1\right] \qquad (4.4.1) \\ &= e\left(\rho(H)\right)\prod_{\beta>0}\left[1-e\left(-\langle\beta,H\rangle\right)\right] \end{aligned}$$

(2) For $\lambda \in \mathrm{t}^*$

$$C_\lambda(H) = \sum_{w\in W}\operatorname{sgn}(w)e\left(\langle\lambda, wH\rangle\right).$$

LEMMA 240. *Both functions are indeed alternating. Further*

(1) *δ vanishes exactly on the singular set.*
(2) *$C_\lambda = 0$ iff λ lies on a wall of a dual chamber.*

PROOF. Clearly C_λ is alternating; for δ let $\alpha \in \Delta$ be a simple root. Then

$$\begin{aligned} \delta(s_\alpha H) &= \prod_{\beta\in\Phi^+}\left[e\left(\frac{1}{2}\langle s_\alpha\beta, H\rangle\right) - e\left(-\frac{1}{2}\langle s_\alpha\beta, H\rangle\right)\right] \\ &= \left[e\left(\frac{1}{2}\langle-\alpha, H\rangle\right) - e\left(-\frac{1}{2}\langle-\alpha, H\rangle\right)\right]\prod_{\beta\in\Phi^+\setminus\{\alpha\}}\left[e\left(\frac{1}{2}\langle s_\alpha\beta, H\rangle\right) - e\left(-\frac{1}{2}\langle s_\alpha\beta, H\rangle\right)\right] \\ &= -\delta(H) = \operatorname{sgn}(s_\alpha)\delta(H), \end{aligned}$$

and the claim follows since W is generated by the simple roots.

Furthermore $\delta(H) = 0$ iff $e\left(\beta(H)\right) = 1$ for some β, that iff $\beta(H) \in \mathbb{Z}$ for some β. For C_λ recall that the group $\mathrm{Stab}_W(\lambda)$ is generated by the root reflections it contains, which are reflections in the walls containing λ. If the stabilizer is trivial $C_\lambda \neq 0$ since exponentials with distinct frequencies are linearly independent; if λ is fixed by some reflection s_β we have

$$C_\lambda = -C_{s_\beta\lambda} = -C_\lambda$$

so $C_\lambda = 0$. □

The function δ is closely connected with the Weyl integration formula.

LEMMA 241. *We have* $\left(\delta\cdot\bar{\delta}\right)(H) = J(\exp H)$.

PROOF. By Equation (4.4.1)

$$\left(\delta\cdot\bar{\delta}\right)(H) = \prod_{\beta\in\Phi}\left[1-e\left(\langle\beta,H\rangle\right)\right].$$

Now the eigenvalues of $\mathrm{Id} - \mathrm{Ad}_{\exp(-H)}$ acting on $(\mathfrak{g}/\mathfrak{t})_{\mathbb{C}}$ are exactly given by the exponential roots $1 - e(\langle \beta, H\rangle)$ and the claim follows. $\square$

PROPOSITION 242. *Let λ be algebraically integral and let*

$$\phi_\lambda = \frac{C_{\rho+\lambda}}{\delta},$$

defined initially on $\mathfrak{g}^{reg}$. Then

(1) *$\phi_\lambda \in R_I$ and thus extends to a symmetric continuous function on $\mathfrak{t}$.*
(2) *If λ is analytically integral ϕ_λ is constant on Λ-cosets, hence extends to a continuous function on T/W.*

PROOF. The function

$$e \circ \rho \cdot C_{\lambda+\rho} = \sum_{w \in W} \mathrm{sgn}(w) e \circ (w\rho + \rho + w\lambda)$$

belongs to R_I since $w\rho + \rho = (w\rho - \rho) + (2\rho) \in \mathbb{Z}[\Delta] \subset \Lambda^*$. It vanishes on the singular set since $C_{\lambda+\rho}$ does (it is alternating!). It follows that

$$\phi_\lambda = \frac{e \circ \rho \cdot C_{\lambda+\rho}}{\prod_{\beta>0}[e \circ \beta - 1]} \in R_I.$$

In the last representation, the denominator is clearly a function on T^{reg}, and if $\lambda \in \Lambda^*$ then the same holds for the numerator by the previous discussion. It follows that ϕ_λ extends to T. $\square$

LEMMA 243. $\mathrm{Span}_{\mathbb{Z}}\{\phi_\lambda\}_{\lambda \in I \cap \mathcal{C}} = R_I^W$ *and similarly for Λ^*.*

PROOF. The functions $d_\lambda = \sum_{w \in W} e \circ \lambda \circ w$ as λ runs over $I \cap \mathcal{C}$ clearly span R_I^W. From the representation

$$\phi_\lambda = \pm \frac{e(-\rho) C_{\lambda+\rho}}{\prod_{\beta>0}[1 - e(-\beta)]}$$

and working in the ring of infinite formal sums $\sum_{\mu \in I} n_\mu \mu$ supposrted on sets of the form $\lambda - \sum_{i=1}^r \mathbb{Z}_{\geq 0} \alpha_i$ (observe that $[1 - e(-\beta)]^{-1} = \sum_{m=0}^\infty e(-m\beta)$, for example) we see that the highest weight occuring in ϕ_λ is λ, where it occurs with multiplicity 1. It follows that for dominant λ we have

$$d_\lambda - \phi_\lambda \in \mathrm{Span}_{\mathbb{Z}}\left\{d_\mu \mid \lambda > \mu \text{ dominant}\right\}$$

and the claim follows by induction. $\square$

4.4.3. The Weyl character formula.

THEOREM 244 (Weyl character formula). *Let λ be an algebraically integral dominant weight. Then for $H \in \mathrm{Lie}\, T$ we have*

$$\chi_\lambda(H) \overset{def}{=} \mathrm{Tr}\left(\exp L^\lambda(H)\right) = \phi_\lambda(H).$$

REMARK 245. If L_λ integrates to a representation π_λ of G then $\exp d\pi_\lambda(H) = \pi_\lambda(\exp H)$ so we have computed the character of π_λ.

EXAMPLE 246. $e(-\rho)\prod_{\beta>0}[e(\beta) - 1] = C_\rho$.

PROOF. The representation with highest weight 0 is the trivial representation, for which the character is identically 1. □

COROLLARY 247 (Weyl dimension formula).

$$\dim L_\lambda = \prod_{\beta>0} \frac{\langle \beta, \lambda+\rho\rangle}{\langle \beta,\rho\rangle} = \prod_{\beta>0} \frac{(\lambda+\rho)(\check{\beta})}{\rho(\check{\beta})}.$$

PROOF. Write the character formula as

$$C_\rho \chi_\lambda = C_{\rho+\lambda}$$

For $H \in \mathfrak{t}$ define the *formal derivative* ∂_H of $e(\mu)$ to be $2\pi i \mu(H) e(\mu)$. This is a derivation of $\mathbb{C}\otimes R_I$ since it's the usual derivative as a function on $\mathfrak{t}$. If we apply any combination $\prod_{i=1}^m \partial_{H_i}$ with $m < \#\Phi^+$ to C_ρ at least one of the factors $e(\beta)-1$ would remain and hence the derivative would vanish at zero. Applying $\prod_{\alpha>0}\partial_{\check{\alpha}}$ to both sides and evaluating at $H=0$, on the LHS we get zero unless we only differentiate C_ρ so

$$\left(\left(\prod_{\beta>0}\partial_{\check{\beta}}\right)\cdot C_\rho\right)(0)\cdot \dim L_\lambda = \left(\left(\prod_{\beta>0}\partial_{\check{\beta}}\right)\cdot C_{\rho+\lambda}\right)(0).$$

Now for any μ

$$\begin{aligned}\left(\left(\prod_{\beta>0}\partial_{\check{\beta}}\right) C_\mu\right)(0) &= \left(\left(\prod_{\beta>0}\partial_{\check{\beta}}\right)\sum_{w\in W}\operatorname{sgn}(w) e(w\mu)\right)(0)\\ &= \sum_w \operatorname{sgn}(w)\prod_{\beta>0}\left\langle \check{\beta}, w\mu\right\rangle\\ &= \sum_w \operatorname{sgn}(w^{-1})\prod_{\beta>0}\left\langle w^{-1}\check{\beta},\mu\right\rangle.\end{aligned}$$

Observe that for reflection in a simple root α we have

$$\begin{aligned}\prod_{\beta>0}\left\langle s_\alpha\check{\beta},\mu\right\rangle &= \langle -\check{\alpha},\mu\rangle \prod_{\beta\in\Phi^+\setminus\{\alpha\}}\left\langle\check{\beta},\mu\right\rangle\\ &= \operatorname{sgn}(s_\alpha)\prod_{\beta>0}\left\langle\check{\beta},\mu\right\rangle.\end{aligned}$$

It follows by induction that the same is true for any product of simple roots, so

$$\left(\left(\prod_{\beta>0}\partial_{\check{\beta}}\right) C_\mu\right)(0) = \#W\prod_{\beta>0}\left\langle\check{\beta},\mu\right\rangle.$$

Dividing the values for $C_{\rho+\lambda}$ and C_ρ now gives the claim. □

Let λ now be an *integral* dominant weight, and let $\phi_\lambda(t)$ be the function on T/W defined by the Weyl character formula. By Corollary 131 the function ϕ_λ extends to a class function on G.

PROPOSITION 248. *The extensions $\{\phi_\lambda\}_{\lambda\in\Lambda^*\cap C}$ are a complete orthonormal system in the space of class functions on G.*

PROOF. By the Weyl integration formula and Lemma 241 we have for integral dominant weights μ,ν that

$$\left\langle \phi_\mu, \phi_\lambda \right\rangle_G = \frac{1}{\#W} \left\langle C_{\rho+\mu}, C_{\rho+\lambda} \right\rangle_T .$$

Now the W-orbits of the characters $\rho+\mu, \rho+\lambda$ are either equal or disjoint (and equal iff $\mu = \lambda$ since a Weyl orbit can intersect the *interior* of a chamber at most once) and the exponential characters are an orthonormal system in $L^2(T)$ (and furthermore for each $w, w' \in W$ the function

$$e\left(-(\rho+\mu)\circ w + (\rho+\lambda)\circ w'\right)$$

is defined on T since $w'\rho - w\rho \in \mathbb{Z}[\Delta]$. It follows that $\{\phi_\lambda\}$ is an orthonormal system of class functions. By Lemma (243) their restrictions to T/W span $R^W_{\Lambda^*}$ which by Stone–Weierstrass/Fourier theory is dense in $C(T/W)$. It follows that $\{\phi_\lambda\}$ are dense in the space of continuous class functions, hence a complete orthonormal system in the space of square-integrable class functions. □

PROOF OF THEOREM 228. Let F be the set of (analytically) integral dominant weights λ so that L_λ integrates to a representation of G. By Theorem 223 this is an enumeration of $\hat{G}$, and by Theorem 244 ϕ_λ are the characters of those representations. By Theorem 36, the characters of all the irreducible representations form a complete orthonormal system in the space of square-integrable class functions. It follows that $\{\phi_\lambda\}_{\lambda\in F} \subset \{\phi_\lambda\}_{\Lambda^*\cap\mathcal{C}}$ are both complete, so $F = \Lambda^* \cap \mathcal{C}$ as claimed. □

CHAPTER 5

Semisimple Lie groups

5.1. Semisimple Lie algebras; the Cartan Involution

5.1.1. Generalities. Let $\mathfrak{g}$ be Lie algebra over a field k of characteristic zero. Recsall the following (PS5):

LEMMA-DEFINITION 249. *The* Killing form $B(X,Y) = \mathrm{Tr}\left(\mathrm{ad}_X \mathrm{ad}_Y \left|\mathfrak{g}\right.\right)$ satisfies:

(1) *It is a symmetric bilinear pairing* $\mathfrak{g}\times\mathfrak{g}\to k$.
(2) *It is* ad*-invariant:* $B(\mathrm{ad}_Z X, Y) + B(X, \mathrm{ad}_Z Y) = 0$ *for all* $X,Y,Z\in\mathfrak{g}$.

LEMMA 250. *Let V be a finite-dimensional k-vector space, $T\in\mathrm{End}_k(V)$, let $U\supset T(V)$ be a subspace containing the image of T. and set $S\colon U\to U$ be the restriction of T to U. Then* $\mathrm{Tr}(T\mid V) = \mathrm{Tr}(S\mid U)$.

PROOF. Extend a basis of U to a basis of V. □

COROLLARY 251. *Let $\mathfrak{a}\subset\mathfrak{g}$ be an ideal. Then for $X,Y\in\mathfrak{a}$ we have $B_{\mathfrak{a}}(X,Y) = B_{\mathfrak{g}}(X,Y)$.*

PROOF. If $X\in\mathfrak{a}$ then the image of $\mathrm{ad}X$ is contained in $\mathfrak{a}$ since for every $Z\in\mathfrak{g}$ we have $\mathrm{ad}_X Z = [X,Z] = -[Z,X]\in\mathfrak{a}$. □

COROLLARY 252. *Let $\mathfrak{a}\subset\mathfrak{g}$ be an abelian ideal. Then $\mathfrak{a}\subset\mathrm{rad}B$.*

PROOF. Let $X\in\mathfrak{g}$ and $Y\in\mathfrak{a}$. Then the image of ad_Y is contained in $\mathfrak{a}$, and ad_X maps $\mathfrak{a}$ to itself. It follows that $B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X\mathrm{ad}_Y|\mathfrak{g}) = \mathrm{Tr}(\mathrm{ad}_X\mathrm{ad}_Y|\mathfrak{a})$. But by hypothesis we have $\mathrm{ad}_Y\restriction_{\mathfrak{a}}= 0$ and it follows that $B(X,Y) = 0$. Since X was arbitrary we conclude that $Y\in\mathrm{rad}B$. □

DEFINITION 253. Call $\mathfrak{g}$ *semisimple* if its Killing form is non-degenerate.

Fix a semisimple Lie algebra $\mathfrak{g}$. For a subset $\mathfrak{a}\subset\mathfrak{g}$ write $\mathfrak{a}^\perp$ for its orthogonal complement with respect to the killing form.

PROPOSITION 254. *Let $\mathfrak{a}\subset\mathfrak{g}$ be an ideal. Then $\mathfrak{a}^\perp$ is an ideal and $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{a}^\perp$ as Lie algebras.*

PROOF. In stages:

(1) $\mathfrak{a}^\perp$ is an ideal: Let $X\in\mathfrak{a}^\perp$, $Z\in\mathfrak{g}$. Then for any $Y\in\mathfrak{a}$, $[Z,Y]\in\mathfrak{a}$ is orthogonal to $X\in\mathfrak{a}^\perp$ and hence
$$B([Z,X],Y) = B(\mathrm{ad}_Z X, Y) = -B(X,\mathrm{ad}_Z Y) = -B(X,[Z,Y]) = 0\,,$$
so that $[Z,X]\in\mathfrak{a}^\perp$.
(2) $\mathfrak{a}$ and $\mathfrak{a}^\perp$ commute: Let $X\in\mathfrak{a}$, $Y\in\mathfrak{a}^\perp$. then for any $Z\in\mathfrak{g}$,
$$B(Z,[X,Y]) = B(X,[Y,Z]) = 0$$
since $[Y,Z]\in\mathfrak{a}^\perp\subset X^\perp$. Since B is non-degenerate we conclude $[X,Y]=0$.

(3) $\mathfrak{a}\cap\mathfrak{a}^{\perp}=\{0\}$: From (1),(2) it follows that $\mathfrak{a}\cap\mathfrak{a}^{\perp}$ is an abelian ideal, and Corollary 252 then shows $\mathfrak{a}\cap\mathfrak{a}^{\perp}\subset\operatorname{rad}B=\{0\}$.

(4) $\mathfrak{a}\oplus\mathfrak{a}^{\perp}=\mathfrak{g}$: From (3) it follows that the restriction of B to $\mathfrak{a}$ is non-degenerate ($\operatorname{rad}B_{\mathfrak{a}}=\mathfrak{a}\cap\mathfrak{a}^{\perp}$). Thus for any $X\in\mathfrak{g}$ there is $Y\in\mathfrak{a}$ such that for all $Z\in\mathfrak{a}$, $B(X,Z)=B(Y,Z)$ (every linear functional on $\mathfrak{a}$ is realized via $B_{\mathfrak{a}}$). Then $X-Y\in\mathfrak{a}^{\perp}$ and thus $X\in\mathfrak{a}+\mathfrak{a}^{\perp}$.

□

COROLLARY 255. *Every semisimple Lie algebra is the direct sum of simple ideals.*

5.1.2. Real semisimple Lie algebras and groups. Fix a real semisimple Lie algebra $\mathfrak{g}$. Realizing $\operatorname{Aut}(\mathfrak{g})\subset\operatorname{GL}(\mathfrak{g})$ as a closed subgroup, we can identify $\operatorname{Lie}(\operatorname{Aut}(\mathfrak{g}))$ with a subalgebra of $\mathfrak{gl}(\mathfrak{g})=\operatorname{End}_{\mathbb{R}}(\mathfrak{g})$. Every ad_X exponentiates in $\operatorname{GL}(\mathfrak{g})$ to an automorphism, so $\operatorname{ad}_X\in\operatorname{Lie}(\operatorname{Aut}(\mathfrak{g}))$

PROPOSITION 256. $\operatorname{Lie}(\operatorname{Aut}(\mathfrak{g})$ *is the image of* $\mathfrak{g}$ *by the adjoint representation.*

PROOF. Write $\mathfrak{h}=\operatorname{Lie}(\operatorname{Aut}(\mathfrak{g})$. Since $Z_{\mathfrak{g}}=\{0\}$, the image of the adjoint map is isomorphic to $\mathfrak{g}$. Next, for $X\in\mathfrak{g}$, $Y\in\mathfrak{h}$ we have $\operatorname{ad}_{[Y,X]}=[Y,\operatorname{ad}X]$ (the second is the commutator in $\mathfrak{gl}(\mathfrak{g})$), and it follows that $\operatorname{ad}\mathfrak{g}\subset\mathfrak{h}$ is a Lie ideal. Its orthogonal complement (wrt the Killing form of $\mathfrak{h}$) is also an ideal since the proof of part (1) of Proposition 254 did not use semisimplicity. The intersection $\operatorname{ad}\mathfrak{g}\cap(\operatorname{ad}\mathfrak{g})^{\perp}$ is then contained in the radical of the restriction of $B_{\mathfrak{h}}$ to $\operatorname{ad}\mathfrak{g}$. But that restriction is $B_{\mathfrak{g}}$ which is non-degenerate, and we conclude that $(\operatorname{ad}\mathfrak{g})^{\perp}\cap\operatorname{ad}\mathfrak{g}=\{0\}$. The proof of part (4) of Proposition 254 now shows that $\mathfrak{h}=\operatorname{ad}\mathfrak{g}\oplus(\operatorname{ad}\mathfrak{g})^{\perp}$ as vector spaces. Finally, the trivial intersection also shows that the two ideals commute: $\left[(\operatorname{ad}\mathfrak{g})^{\perp},\operatorname{ad}\mathfrak{g}\right]\subset(\operatorname{ad}\mathfrak{g})^{\perp}\cap\operatorname{ad}\mathfrak{g}=\{0\}$, which means that any $D\in(\operatorname{ad}\mathfrak{g})^{\perp}$ acts trivially on $\mathfrak{g}$. But since these are derivations of $\mathfrak{g}$ it follows that $D=\{0\}$ – in other words that $(\operatorname{ad}\mathfrak{g})^{\perp}=0$ and $\mathfrak{h}=\operatorname{ad}\mathfrak{g}$. □

DEFINITION 257. A Lie group G is *semisimple* if its Lie algebra is.

COROLLARY 258. *Let G be a semisimple Lie group. then* $\operatorname{Ad}(G)$ *is the connected component of* $\operatorname{Aut}(\mathfrak{g})$ *and in particular is closed in* $\operatorname{Aut}(\mathfrak{g})$ *and hence* $\operatorname{GL}(\mathfrak{g})$.

COROLLARY 259 (Converse to PS5 Problem 6(e)). *Let G be a connected Lie group with finite centre and suppose that its Killing form is negative definite. Then G is compact.*

PROOF. G is semisimple since its Killing form is non-degenerate. $G/Z(G)=\operatorname{Ad}(G)$ is a closed subgroup of $\operatorname{GL}(\mathfrak{g})$ preserving a definite quadratic form, hence compact. □

LEMMA 260. *Let G be semisimple. Then* $\operatorname{Ad}(G)$ *is centerfree.*

PROOF. Suppose $\operatorname{Ad}_g\in\operatorname{Ad}(G)$ is central. Then for any $X\in\mathfrak{g}$and $t>0$ we have

$$\exp(t\operatorname{ad}(\operatorname{Ad}_gX))=\exp\left(t\operatorname{Ad}_g\operatorname{ad}X\operatorname{Ad}_{g^{-1}}\right)=\operatorname{Ad}_g\exp(t\operatorname{ad}X)\operatorname{Ad}_{g^{-1}}=\operatorname{Ad}(\exp(tX))=\exp(t\operatorname{ad}X)$$

Differentiating wrt t we have $\operatorname{ad}(\operatorname{Ad}_gX)=\operatorname{ad}X$. Since $\operatorname{ad}\mathfrak{g}\simeq\mathfrak{g}$ we conclude that $\operatorname{Ad}_gX=X$ so that $\operatorname{Ad}_g=\operatorname{id}$. □

- Do not discuss in class:

THEOREM 261 (Jordan decomposition). *Let $X\in\mathfrak{g}$. Then there exist commuting $X_s,X_n\in\mathfrak{g}$ such that ad_{X_s} is semisimple, ad_{X_n} is nilpotent and $X=X_s+X_n$.*

PROOF. $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ is an algebraic group with Lie algebra $\mathrm{ad}\,\mathfrak{g} \simeq \mathfrak{g}$, and the claim now follows from the corresponding claim for algebraic groups (see 4.4[**2**, Thm.\{} 2.4.8, Thm.\{} 4.4.20,]). ☐

5.1.3. Real forms and Cartan involutions. Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification. Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ be a Cartan subalgebra. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \bigoplus_{\alpha\in\Phi(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h})} \mathfrak{g}_{\alpha}\,.$$

FACT 262. *Can choose basis $X_\alpha \in \mathfrak{g}_\alpha$ so that*

(1) $[X_\alpha, X_{-\alpha}] = H_\alpha \in \mathfrak{h}$ *is the coroot,*
(2) *If $\alpha+\beta$ is root then $\left[X_\alpha, X_\beta\right] = N_{\alpha,\beta}X_{\alpha+\beta}$ for real $N_{\alpha,\beta}$ such that $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$.*

COROLLARY 263. *Let $\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \forall\alpha\in\Phi : \alpha(H) \in \mathbb{R}\} = \mathrm{Span}_{\mathbb{R}}\{H_\alpha\}_{\alpha\in\Phi}$. Then*

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathbb{R}X_\alpha$$

is a real form of $\mathfrak{g}$, the split real form.

COROLLARY 264. *Let*

$$\mathfrak{u} = i\mathfrak{h}_0 \oplus \bigoplus_{\alpha>0} \mathbb{R}\left(X_\alpha - X_{-\alpha}\right) \oplus \bigoplus_{\alpha>0} i\mathbb{R}\left(X_\alpha + X_{-\alpha}\right)$$

is a real form with negative-definite killing form, hence a compact real form.

PROOF. Direct computation. ☐

LEMMA 265. *For $X \in \mathfrak{g}_{\mathbb{C}}$ let $\tau(X)$ denote complex conjugation wrt $\mathfrak{u}$. Then $\tau([X,Y]) = [\tau(X), \tau(Y)]$. Furthermore, let $\tilde{B}$ be the killing form of $\mathfrak{g}_{\mathbb{C}}$ considered as a **real*** Lie algebra. Then $\tilde{B}(X, \tau(Y))$ is negative-definite.

DEFINITION 266. Let $\mathfrak{g}$ be a real Lie algebra. An involution $\theta \in \mathrm{Aut}(\mathfrak{g})$ such that $B_\theta(X,Y) = B_{\mathfrak{g}}(X, \theta Y)$ is negative-definite is called a *Cartan involution*.

EXAMPLE 267. $\mathfrak{g} = \mathfrak{gl}_n\mathbb{R}$, $\theta(X) = -X^*$ (negative transpose).

- Every complex semisimple Lie algebra has a Cartan involution.

LEMMA 268. *Let $g \in \mathrm{Aut}(\mathfrak{g})$ be symmetric positive definite with respect to B_θ. Then $g = \exp\mathrm{ad}X$ for some $X \in \mathfrak{g}$ (in particular $g \in \mathrm{Aut}(\mathfrak{g})^\circ$)*

PROOF. Let $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ be the spectral decomposition of $\mathfrak{g}$ wrt g. As usual $\left[\mathfrak{g}_\lambda, \mathfrak{g}_\mu\right] = \mathfrak{g}_{\lambda\mu}$. Thus for any $t > 0$ if $X \in \mathfrak{g}_\lambda$, $Y \in \mathfrak{g}_\mu$ we have

$$\left[g^t X, g^t Y\right] = (\lambda\mu)^t [X,Y] = g^t [X,Y]$$

and it follows that $g^t \in \mathrm{Aut}(\mathfrak{g})$. The one-parameter subgroup $\{g^r\}_{r>0} \subset \mathrm{Aut}(\mathfrak{g})$ is then of the form $\exp(tX)$ with $X \in \mathrm{Lie}\left(\mathrm{Aut}(\mathfrak{g})\right) = \mathrm{ad}\,\mathfrak{g}$. ☐

THEOREM 269. *Every real semisimple Lie algebra has a Cartan involution.*

PROOF. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Let $\mathfrak{h} = \mathfrak{g} \oplus_{\mathbb{R}} i\mathfrak{g}$ viewed as a *real* Lie algebra. By Lemma 265 $\mathfrak{h}$ is real semisimple with a Cartan involuion θ; in addition $\mathfrak{g}$is the set of fixed points of the involution $\sigma \in \mathrm{Aut}(\mathfrak{h})$ given by complex conjugation. Then $\rho = \sigma\theta$ is an automorphism of $\mathfrak{h}$. For $X, Y \in \mathfrak{h}$ we have

$$\begin{aligned} B_{\mathfrak{h}}(\rho X, \theta Y) &= B_{\mathfrak{h}}\left(X, \rho^{-1}\theta Y\right) \\ &= B_{\mathfrak{h}}(X, \theta\sigma\theta Y) \\ &= B_{\mathfrak{h}}(X, \theta(\rho Y)) \,. \end{aligned}$$

It follows that ρ is symmetric wrt B_θ and hence that ρ^2 is symmetric positive-definite. We have $\rho^2\theta\rho^2 = \sigma\theta\sigma\theta\theta\sigma\theta\sigma\theta = \theta$. Working in a basis where ρ is diagonal we get $|\rho|^t \theta |\rho|^t = \theta$ for all t (and that $|\rho|$ and ρ commute).

Consider now the Cartan involution $\tilde{\theta} = |\rho|^{1/2} \theta |\rho|^{-1/2}$. We have

$$\begin{aligned} \tilde{\theta}\sigma &= \left(|\rho|^{1/2} \theta |\rho|^{-1/2}\right)\sigma \\ &|\rho| \left(|\rho|^{-1/2} \theta |\rho|^{-1/2}\right)\sigma \\ &= |\rho| \theta\sigma = |\rho| \rho^{-1} \end{aligned}$$

and similarly that

$$\sigma\tilde{\theta} = \sigma |\rho| \theta = \sigma\theta |\rho|^{-1} = \rho |\rho|^{-1} \,.$$

Now working in a basis where ρ is diagonal we see that $|\rho| \rho^{-1} = \rho |\rho|^{-1}$ that is $\sigma\tilde{\theta} = \tilde{\theta}\sigma$. It follows that $\tilde{\theta}$ acts on the fixed-point set of σ, that is on $\mathfrak{g}$. It remains to show that $\tilde{\theta} \restriction_{\mathfrak{g}}$ is a Cartan involution. For $X, Y \in \mathfrak{g}$ we have that $\mathrm{ad}_{\mathfrak{h}} X$, $\mathrm{ad}_{\mathfrak{h}} Y$ are block-diagonal with respect to the decomposition $\mathfrak{h} = \mathfrak{g} \oplus_{\mathbb{R}} i\mathfrak{g}$ with equal blocks, so $B_{\mathfrak{h}}(X,Y) = 2B_{\mathfrak{g}}(X,Y)$ and so

$$B_{\mathfrak{g}}(X, \tilde{\theta}Y) = \frac{1}{2} B_{\mathfrak{h}}(X, \tilde{\theta}Y)$$

is negative-definite. □

5.2. Cartan and Iwasawa Decompositions

Fix a Cartan involution θ of $\mathfrak{g}$. Let $\mathfrak{k}, \mathfrak{p}$ be the $+1, -1$ eigenspaces respectively. Then $\mathfrak{k}$ is a subalgebra and $\mathfrak{p}$ is a $\mathfrak{k}$-module, so they are orthogonal wrt $B_{\mathfrak{g}}$ and B_θ. For $X, Y \in \mathfrak{k}$ it follows that $B_{\mathfrak{k}}(X,Y) = B_{\mathfrak{g}}(X,Y) = B_{\mathfrak{g}}(X, \theta Y)$ is negative definite, so $\mathfrak{k}$ is a compact Lie algebra.

OBSERVATION 270. *$\mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$.*

LEMMA 271. *Let $*$ denote adjoints wrt to the inner product B_θ. Then $(\mathrm{ad} X)^* = -\mathrm{ad}(\theta X)$.*

COROLLARY 272. *$\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ is a subalgebra closed under transpose.*

THEOREM 273. *Let G be connected semisimple, θ a Cartan involution of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

1. *There exists an involution $\Theta \in \mathrm{Aut}(G)$ with $d\Theta = \theta$.*
2. *$G^\Theta = K$ is the subgroup with Lie algebra $\mathfrak{k}$. It is closed, contains $Z = Z(G)$ and is compact mod centre.*
3. *(Cartan decomposition) The map $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism.*
4. *When Z is finite K is a maximal compact subgroup.*

PROOF. Consider first the case of $\bar{G} = \mathrm{Ad}(G)$. Equipping $\mathfrak{g}$ with the inner product B_θ if $g \in \mathrm{Aut}(\mathfrak{g})$ then for any X, Y we have $\left[gXg^{-1}, gYg^{-1}\right] = [X,Y]$ as linear maps in $\mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ (idenfying X with $\mathrm{ad}X$). Recalling that $[X,Y] = XY - YX$ and taking transposes we find

$$-\left[(g^{-1})^* X^* g^*, (g^{-1})^* Y^* g^*\right] = -\left[X^*, Y^*\right].$$

Since $\mathfrak{g}$ is closed under transpose, we see that $(g^*)^{-1} \in \mathrm{Aut}(\mathfrak{g})$ so $g^* \in \mathrm{Aut}(\mathfrak{g})$ as well. Define now $\bar{\Theta}(g) = (g^*)^{-1} \in \mathrm{Aut}(\bar{G})$. Clearly $\bar{\Theta}^2 = \mathrm{Id}$. Its differential is $X \mapsto -X^*$ which is θ.

The fixed points of $\bar{\Theta}$ are the compact subgroup $\bar{K} = \mathrm{Aut}(\mathfrak{g}) \cap O(B_\theta)$ with Lie algebra $\{X \in \mathfrak{g} \mid X^* = -X\} = \mathfrak{k}$.

The map $k \exp X$ is smooth $\bar{\Theta} \times \mathfrak{p} \to \bar{G}$. We first construct the inverse: suppose $g = k \exp X$. then $g^* g = \exp(X^*) k^* k \exp(X) = \exp(2X)$ and X is uniquely determined as the logarithm of a positive-definite matrix (and moreover the map $g \to X$ is smooth), at which point $k = g \exp(-X)$ shows that k is also unique and depends smoothly on g.

To see that the inverse is defined everywhere note that for $g \in G$ the element $g^* g \in G$ is positive-definite, hence of the form $\exp 2X$ for some $X \in \mathfrak{g}$. Since $\exp(2tX) = (g^* g)^t$ are all positive-definite, the same is true for the derivative at $t = 0$, so $X \in \mathfrak{p}$. Let $k = g \exp(-X)$. Then

$$k^* k = \exp(-X)\, g^* g \exp(-X) = \mathrm{Id}_{\mathfrak{g}}$$

so $k \in K$ and $g = k \exp(X)$.

Returning to the general case, since G is semisimple $Z_{\mathfrak{g}} = \{0\}$ so Z is discrete, and $\mathrm{Ad}(G)$ is the covering map $G \mapsto G/Z$. Let K be the inverse image of $\bar{K}$ under the quotient map, necessarily a closed subgroup which is compact mod Z. Let $q\colon G \to \bar{G}$ be the covering map Ad, and consider its extension $\tilde{q}\colon G/K \to \bar{G}/\bar{K}$. The map is continuous and surjective since q is, and injective since $Z \subset K$. Suppose now that

$$q(g_n)\bar{K} \to q(g)\bar{K}$$

in $\bar{G}/\bar{K}$. We want to prove that $g_n K \to gK$ in G/K. Since $\bar{K}$ is compact we can assume wlog that $q(g_n) \to q(g)$. Then $q(g^{-1}g_n) \to 1_{\bar{G}}$. By the covering property we have an open neighbourhood $1 \in \bar{U} \subset \bar{G}$ such that $q^{-1}(\bar{U}) = U \times Z$ for a neighbourhood $1 \in U \subset G$. Write $g_n = u_n z_n$ and $g = uz$. Then $g^{-1} g_n = (z^{-1} z_n)(u^{-1} u_n)$ with $u^{-1} u_n \to 1$. It follows that $u_n Z K \to uZK$ i.e. that $g_n K \to gK$.

For $g \in G$ we have $q(g) = q(k) q(\exp X)$ for some $k \in K$, $X \in \mathfrak{p}$ and then $g = kz \exp(X)$ for some $z \in Z$, so $g = (kz)\exp X \in K \exp \mathfrak{p}$. The uniqueness of X follows from the uniqueness in $\bar{G}$, and this gives the uniqueness of kz as well. Since q is a local diffeomorphism and since the Cartan decomposition of $\bar{G}$ is a diffeomorphism, the same is true for G. Thus K is a smooth deformation retract of G. Thus makes K connected, hence the Lie subgroup with Lie algebra $\mathfrak{k}$, and in addition $\pi_1(G) \simeq \pi_1(K)$.

Finally let $\tilde{G} \to G \to \bar{G}$ be the universal covering group. Everything thus far applies to $\tilde{G}$, giving a connected subgroup $\tilde{K}$ such that $\tilde{K}$ covering K. The Lie algebra homomorphism θ extends to a Lie group homomorphism $\tilde{\Theta}$ of $\tilde{G}$; since θ is trivial on $\mathfrak{k}$ its exponential $\tilde{\Theta}$ is trivial on $\tilde{K}$ and in particular on $\tilde{Z}$. It follows that $\tilde{\Theta}$ descends to an involution Θ of G with fixed point set K. □

Let A, N be the subgroups correpsonding to $\mathfrak{a}$, $\mathfrak{n}$ and let $M = Z_K(\mathfrak{a}) = Z_K(A)$.

PROPOSITION 274. *A is a closed subgroup and* $\exp\colon \mathfrak{a} \to A$ *is a diffeomorphism.*

PROOF. We first verify that $\exp\colon \mathfrak{a} \to A$ is proper. Indeed, $\Delta \subset \mathfrak{a}_{\mathbb{R}}^*$ is a basis, so $\|H\|_\infty = \max_{\alpha \in \Phi} |\alpha(H)|$ is a norm on $\mathfrak{a}$. Since $\exp(\alpha(H))$ are all eigenvalues of $\mathrm{Ad}_{\exp H}$ it follows that for any operator norm on $\mathrm{End}(\mathfrak{g})$,

$$\left\|\mathrm{Ad}_{\exp H}\right\| \geq \exp\|H\|_{\infty} .$$

By Theorem 112,exp: $\mathfrak{a} \to A$ is a covering map. Being proper its kernel is compact hence trivial. Also, since $\mathfrak{a}$ is locally compact its image by a proper map is closed. $\square$

REMARK 275. When G is reductive we have $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{z}$ with $\mathfrak{a}_1$ the Cartan subgroup of the derived group. The argument above applies to $A_1 = \exp \mathfrak{a}_1$; for $\mathfrak{z}$ it's enough to note that $Z(G)$ is closed.

PROPOSITION 276. *N is a closed subgroup and* $\exp\colon \mathfrak{n} \to N$ *is a diffeomorphism.*

PROOF. We prove the second claim first. The exponential map is always a local diffeomorphism, and the only question is whether it is bijective, and surjectivity holds since $\mathfrak{n}$ is nilpotent. Suppose that $\exp X = \exp Y$ for some $X, Y \in \mathfrak{n}$. We can write $X = \sum_{\alpha>0} X_\alpha$, $Y = \sum_\alpha Y_\alpha$ with $X_\alpha, Y_\alpha \in \mathfrak{g}_\alpha$. Choose $H \in \mathfrak{a}$ in the interior of the positive Weyl chamber, in other words such that $\alpha(H) > 0$ for all $\alpha > 0$. Then $\mathrm{Ad}_{\exp(-tH)} \exp(X) = \exp\left(\sum_{\alpha>0} \exp(-t\alpha(H) X_\alpha\right)$ so $\exp\left(\sum_{\alpha>0} \exp(-t\alpha(H) X_\alpha\right) = \exp\left(\sum_{\alpha>0} \exp(-t\alpha(H) Y_\alpha\right)$. Thus if t is large enough the local injectivity of the exponential map gives $\sum_{\alpha>0} \exp(-t\alpha(H) X_\alpha = \sum_{\alpha>0} \exp(-t\alpha(H) Y_\alpha$ from which we conclude for each α that $\exp(-t\alpha(H) X_\alpha = \exp(-t\alpha(H) Y_\alpha$ and $X_\alpha = Y_\alpha$, hence that the exponential map is injective.

To see that N is closed, let $\overline{N}$ denotes its topological closure. This is a connected nilpotent subgroup normalized by A, so $\mathrm{Lie}(\overline{N}) = \left(\mathrm{Lie}(\overline{N}) \cap \mathfrak{g}_0\right) \oplus \bigoplus_\alpha \left(\mathrm{Lie}(\overline{N}) \cap \mathfrak{g}_\alpha\right)$. $\mathrm{Lie}(\overline{N})$ certainly contains $\mathfrak{n}$. It cannot contain any element of $\mathfrak{a}$ (this would give it a solvable but non-nilpotent subalgebra) or of $\mathfrak{g}_\alpha$ for $\alpha < 0$ (this would induce an element of $\mathfrak{a}$). It follows that $\mathfrak{n} \subset \mathrm{Lie}(\bar{N}) \subset \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m}$ so if if the first containment is proper $\mathrm{Lie}(\bar{N})$ contains a nonzero element of $\mathfrak{a} \oplus \mathfrak{m}$, that is an ad-semisimple element. But and the set of nilpotent element of $\mathrm{End}(\mathfrak{g})$ is closed. Since every non-zero element of $\mathfrak{a} \oplus \mathfrak{m}$ is *ad*-semisimple and normalizes $\mathfrak{n}$ it follows that $\mathrm{Lie}(\bar{N}) = \mathfrak{n}$ and hence that $\bar{N} = N$. $\square$

COROLLARY 277. *The semidirect product NA is a closed subgroup diffeomorphic to its Lie algebra* $\mathfrak{n} \oplus \mathfrak{a}$.

LEMMA 278. *Let G be a Lie group, $S, T < G$ two subgroups with Lie algebras* $\mathfrak{s}, \mathfrak{t}$ *such that* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$. *Then the map $m\colon S \times T \to G$ given by $(s,t) \mapsto st$ is everywhere regular, and in particular open.*

PROOF. We compute the derviative. For $X \in \mathfrak{s}$, $Y \in \mathfrak{t}$ we have $s\exp(X)t\exp(Y) = s\exp(X)\exp(\mathrm{Ad}_t Y)t$. It follows that $dm_{(s,t)}(s_* X, t_* Y) = (s\cdot)_* \left(X + \mathrm{Ad}_t Y\right)(\cdot t)_*$. Now $\mathrm{Ad}_t Y \in \mathfrak{t}$ as well, so $X + \mathrm{Ad}_t Y = 0$ iff $X = \mathrm{Ad}_t Y = 0$ iff $X = Y = 0$ and the map is injective. $\square$

THEOREM 279. *The multiplication map $m\colon N \times A \times K \to G$ is a diffeomorphism onto.*

PROOF. We already know this for the map $N \times A \to NA$. Applying the Lemma to the multiplication map $NA \times K \to G$ we see that it is $\square$

CHAPTER 6

Representation theory of real groups

APPENDIX A

Functional Analysis

In this appendix we review the basics of topological vector spaces. References include Schaefer–Wolff.

A.1. Topological vector spaces

Let K be a non-discrete complete valued field

DEFINITION 280. A *topological vector space* is a vector space V over K equipped with a topology so that $(V,+)$ is a topological group and such that scalar multiplication is a continuous map $\cdot : K \times V \to V$.

PROPOSITION 281. *A finite-dimensional K-vector space has a unique topology making it into a TVS. In particular, if V,W are TVS with V finite-dimensional then* $\mathrm{Hom}_K(V,W) = \mathrm{Hom}_{cts}(V,W)$ *and if $V \subset W$ then V is closed and complete. If K is locally compact then a TVS over K is locally compact iff it is finite-dimensional.*

DEFINITION 282. *Fix a TVS V. Call $C \subset V$:*

(1) *Balanced*, if $\alpha \underline{v} \in C$ for all $x \in C$, $|\alpha| \leq 1$
(2) *Absorbing*, if $\cup_{t>0} tC = V$ (that is, for all $\underline{v} \in V$ there are $\underline{u} \in C$ and $t > 0$ such that $t\underline{u} = \underline{v}$.
(3) *Bounded*, if for every open neighbourhood $W \ni \underline{0}$ there is $t > 0$ such that $C \subset tW$.
(4) *Totally bounded,* if for every open neighbourhood $W \ni \underline{0}$ there is a finite set $\{\underline{u}_i\}_{i=1}^n \subset V$ such that $C \subset \cup_i (\underline{v}_i + W)$.

LEMMA 283. *Every finite subset of a TVS is bounded.*

LEMMA 284. *Every TVS has a basis neighbourhoods of $\underline{0}$ which are balanced.*

DEFINITION 285. A net $\{x_\alpha\}_{\alpha \in D} \subset V$ is called a *Cauchy net* if for every neighbourhood W of 0 there is $\delta \in D$ such that if $\alpha, \beta \geq \delta$ then $x_\alpha - x_\beta \in W$. $X \subset V$ is *complete* if every Cauchy net in X converges to a limit in X. V is *quasi-complete* if every closed bounded subset of X is complete.

LEMMA 286. *In a quasi-complete TVS every totally bounded subset is relatively compact.*

ASSUMPTION 287. $K = \mathbb{R}$ *or* $\mathbb{C}$.

DEFINITION 288. *Fix a TVS V. Call $C \subset V$ convex*, if $t\underline{u} + (1-t)\underline{v} \in C$ for all $\underline{u}, \underline{v} \in C$, $t \in [0,1]$. Call V locally convex *if any neighbourhood of* 0 *contains a convex neighbourhood of zero.*

PROPOSITION 289. *A TVS is locally convex iff its topology is determined by a family of seminorms.*

LEMMA 290. *Let V be locally convex, $C \subset V$ be totally bounded. Then the convex hull and balanced convex hull of C are also totally bounded.*

COROLLARY 291. *Let V be locally convex and quasi-complete and let $C \subset V$ be compact. Then the closed convex hull of C is compact.*

DEFINITION 292. The *continuous dual* of V is $V' \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{cts}}(V,K)$.

THEOREM 293 (Hahn–Banach). *Let V be locally convex, $U \subset E$ a subspace, $f \in U'$. Then f has a continuous linear extension to V. In particular, V' separates the points of V.*

A.2. Quasicomplete locally convex TVS

[based on Casseleman, Garrett]

PROPOSITION 294. *An inverse limit of quasi-complete spaces is quasi-complete. The direct product of a family of quasi-complete space is quasi-complete. The weak-* dual of a Banach space is quasi-complete.*

Let V be a locally convex TVS.

DEFINITION 295. Let Ω be a measureable space.

(1) Call $f\colon \Omega \to V$ *weakly measurable* if $\varphi \circ f\colon \Omega \to K$ is measurable for each $\varphi \in V'$. Let
(2) Let μ be a measurae on Ω and let $f\colon \Omega \to V$ be weakly measurable. Call $\underline{v} \in V$ the *Gelfand–Pettis integral* of f (and write $\underline{v} = \int f\,\mathrm{d}\mu$) if for every $\varphi \in V'$ $\varphi \circ f$ is μ-integrable and we have
$$\varphi(\underline{v}) = \int_\Omega \varphi \circ f\,\mathrm{d}\mu\,.$$

REMARK 296. Note that the integral clearly exists as an element of V''; the question is about existence as an element of V. Since V' separates the points, it is also clear that the integral (if it exists) is unique.

THEOREM 297. *Let V be quasi-complete, let Ω be compact, μ a Radon measure, and let $f\colon \Omega \to V$ be continuous. Then $\int f\,\mathrm{d}\mu$ exists.*

PROOF. Wlog μ is a probability measure. In that case we also show $\int f\,\mathrm{d}\mu$ lies in the closed convex hull of $f(\Omega)$.

LEMMA 298. *If V is finite-dimensional then $\int f\,\mathrm{d}\mu$ exists and lies in the convex hull of $f(\Omega)$.*

Write C for the closed convex hull of $f(\Omega)$. For every finite $\mathcal{F} \subset V'$ consider the continuous linear map $\mathcal{F}\colon V \to K^{\mathcal{F}}$ given by $\underline{v} \mapsto (\varphi(\underline{v}))_{\varphi\in\mathcal{F}}$. It maps C continuously onto the convex hull of the image of $\mathcal{F}\circ f$. Now $\int_\Omega (\mathcal{F}\circ f)\,\mathrm{d}\mu$ exists in that convex hull, and we obtain a non-empty closed convex subset
$$C_{\mathcal{F}} = \left\{\underline{v} \in C \mid \mathcal{F}(\underline{v}) = \int_\Omega (\mathcal{F}\circ f)\,\mathrm{d}\mu\right\}.$$
Since $\bigcap_{i=1}^r C_{\mathcal{F}_i} = C_{\bigcup_i \mathcal{F}_i}$ we see that this family has the finite intersection property, and it follows that
$$\bigcap_{\mathcal{F}} C_{\mathcal{F}}$$
is non-empty. The (necessarily unique) point there is the desired integral. □

A.3. Integration

A.4. Spectral theory and compact operators

A.5. Trace-class operators and the simple trace formula

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