Lior Silberman's Math 412: Problem Set 5 (due 17/2/2023)

Practice

P1. Let $U = \mathrm{Span}_F\{\underline{u}_1, \underline{u}_2\}$ be two-dimensional. Show that the element $\underline{u}_1 \otimes \underline{u}_1 + \underline{u}_2 \otimes \underline{u}_2 \in U \otimes U$ is not a pure tensor, that is not of the form $\underline{u} \otimes \underline{v}$ for any $\underline{u}, \underline{v} \in U$.

P2. Let $\iota \colon U \times V \to U \otimes V$ be the standard inclusion map ($\iota(\underline{u}, \underline{v}) = \underline{u} \otimes \underline{v}$). Show that $\iota(\underline{u}, \underline{v}) = 0$ iff $\underline{u} = \underline{0}_U$ or $\underline{v} = \underline{0}_V$ and that for non-zero vectors we have $\iota(\underline{u}, \underline{v}) = \iota(\underline{u}', \underline{v}')$ iff $\underline{u}' = \alpha \underline{u}$ and $\underline{v}' = \alpha^{-1} \underline{v}$ for some $\alpha \in F^\times$.

P3. Let U, V be finite-dimensional spaces and let $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$.
- (a) Construct a map $A \oplus B \in \mathrm{End}_F(U \oplus V)$ restricting to A, B on the images of U, V in $U \oplus V$.
- (b) Show that $\mathrm{Tr}(A \oplus B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$.
- (c) Evaluate $\det(A \oplus B)$.

The trace

1. Let U be finite-dimensional.
   - (a) Construct an isomorphism $V \otimes U' \to \mathrm{Hom}_F(U, V)$.
   - (b) Define a map $\mathrm{Tr} \colon U \otimes U' \to F$ extending the evaluation pairing $U \times U' \to F$.
   - DEF The *trace* of $T \in \mathrm{Hom}_F(U, U)$ is given by identifying T with an element of $U \otimes U'$ via (a) and then applying the map of (b).
   - (c) Let $T \in \mathrm{End}_F(U)$, and let A be the matrix of T with respect to the basis $\{\underline{u}_i\}_{i=1}^n \subset U$. Show that $\mathrm{Tr}\, T = \sum_{i=1}^n A_{ii}$.
   - RMK This shows that similar matrices have the same trace!
   - (d) Solve P3(b) from this point of view.

Tensor products of maps

2. Let U, V be finite-dimensional spaces, and let $A \in \mathrm{End}(U)$, $B \in \mathrm{End}(V)$.
   - (a) Show that $(\underline{u}, \underline{v}) \mapsto (A\underline{u}) \otimes (B\underline{v})$ is bilinear, and obtain a linear map $A \otimes B \in \mathrm{End}(U \otimes V)$.
   - (b) Suppose A, B are diagonable. Using an appropriate basis for $U \otimes V$, Obtain a formula for $\det(A \otimes B)$ in terms of $\det(A)$ and $\det(B)$.
   - (c) Extending (a) by induction, show for any $A \in \mathrm{End}_F(V)$, the map $A^{\otimes k}$ induces maps $\mathrm{Sym}^k A \in \mathrm{End}(\mathrm{Sym}^k V)$ and $\bigwedge^k A \in \mathrm{End}(\bigwedge^k V)$.
   - (*d) Show that the formula of (b) holds for all A, B.

3. Suppose $\frac{1}{2} \in F$, and let U be finite-dimensional. Construct isomorphisms

$$\{\text{ symmetric bilinear forms on } U\} \leftrightarrow \left(\mathrm{Sym}^2 U\right)' \leftrightarrow \mathrm{Sym}^2\left(U'\right).$$

Extension of scalars

4. (extension of scalars for linear maps) Let K/F be an extension of fields. For $T \in \mathrm{Hom}_F(U,V)$ let $T_K = \mathrm{Id}_K \otimes_F T \in \mathrm{Hom}_F(U_K, V_K)$ as in 2(a).
   (a) Show that $T_K \in \mathrm{Hom}_K(U_K, V_K)$ (i.e. that it is actually K-linear not only F-linear).
   (b) (Functoriality) Show that $\mathrm{Id}_{U_K} = (\mathrm{Id}_U)_K$. For $S \in \mathrm{Hom}_F(V,W)$. Show that $(S \circ T)_K = S_K \circ T_K$.
   (c) (Linear algebra) If $U \subset V$ we identify U_K with a subspace of V_K via the inclusion map. Show that (with this identification) we have $\mathrm{Ker}\, T_K = (\mathrm{Ker}\, T)_K$ and $\mathrm{Im}\, T_K = (\mathrm{Im}\, T)_K$.
   (d) Let $B_U \subset U$ be an F-basis. Show that $\{1_K \otimes \underline{u}\}_{\underline{u} \in B_U} \subset U_K$ is a K-basis of U_K.
   (e) Let B_U, B_V be bases of U, V respectively. Show that the matrix of T_K with respect to the corresponding bases of U_K, V_K is the same as the matrix of T with respect to the original bases.

5. (extension of scalars and constructions) Construct "natural" isomorphisms:
   (a) $\bigoplus_{i \in I} (V_i)_K \to (\bigoplus_{i \in I} V_i)_K$
   (b) $U_K / V_K \to (U/V)_K$.
   (c) $U_K \otimes_K V_K \to (U \otimes_F V)_K$.
   HINT in each case show that both sides satisfy the appropriate universal property for K-vectorspaces.
   (*d) Show that the natural map $(\prod_{i \in I} V_i)_K \to \prod_{i \in I} (V_i)_K$ is, in general, not surjective.

Supplement: Nilpotence

A. Let $U \in M_n(F)$ be *strictly upper-triangular*, that is upper triangular with zeroes along the diagonal. Show that $U^n = 0$ and construct such U with $U^{n-1} \neq 0$.

B. Let V be a finite-dimensional vector space, $T \in \mathrm{End}(V)$.
   (*a) Show that the following statements are equivalent:
   (1) $\forall \underline{v} \in V : \exists k \geq 0 : T^k \underline{v} = \underline{0}$; (2) $\exists k \geq 0 : \forall \underline{v} \in V : T^k \underline{v} = \underline{0}$.
   DEF A linear map satisfying (2) is called *nilpotent*. Example: see problem 3.
   SUPP For any infinite-dimensional V find an example of $T \in \mathrm{End}(V)$ satisfying (1) but not (2). Such maps are called *locally nilpotent*.
   (b) Find nilpotent $A, B \in M_2(F)$ such that $A + B$ isn't nilpotent.
   (c) Suppose that $A, B \in \mathrm{End}(V)$ are nilpotent and that A, B commute. Show that $A + B$ is nilpotent.

Supplement: the tensor algebra

C. (The tensor algebra) Fix a vector space U.

(a) Extend the bilinear map $\otimes\colon U^{\otimes n}\times U^{\otimes m}\to U^{\otimes n}\otimes U^{\otimes m}\simeq U^{\otimes(n+m)}$ to a bilinear map $\otimes\colon \bigoplus_{n=0}^{\infty}U^{\otimes n}\times\bigoplus_{n=0}^{\infty}U^{\otimes n}\to\bigoplus_{n=0}^{\infty}U^{\otimes n}$.

(b) Show that this map $\otimes$ is associative and distributive over addition. Show that $1_F\in F\simeq U^{\otimes 0}$ is an identity for this multiplication.

DEF This algebra is called the *tensor algebra* $T(U)$.

(c) Show that the tensor algebra is *free:* for any F-algebra A and any F-linear map $f\colon U\to A$ there is a unique F-algebra homomorphism $\bar{f}\colon T(U)\to A$ whose restriction to $U^{\otimes 1}$ is f.

D. (The symmetric algebra). Fix a vector space U.

(a) Endow $\bigoplus_{n=0}^{\infty}\mathrm{Sym}^n U$ with a product structure as in 3(a).

(b) Show that this creates a commutative algebra $\mathrm{Sym}(U)$.

(c) Fixing a basis $\{\underline{u}_i\}_{i\in I}\subset U$, construct an isomorphism $F\left[\{x_i\}_{i\in I}\right]\to\mathrm{Sym}^* U$.

RMK In particular, $\mathrm{Sym}^*(U')$ gives a coordinate-free notion of "polynomial function on U".

(d) Let $I\lhd T(U)$ be the two-sided ideal generated by all elements of the form $\underline{u}\otimes\underline{v}-\underline{v}\otimes\underline{u}\in U^{\otimes 2}$. Show that the map $\mathrm{Sym}(U)\to T(U)/I$ is an isomorphism.

RMK When the field F has finite characteristic, the correct definition of the symmetric algebra (the definition which gives the universal property) is $\mathrm{Sym}(U)\stackrel{\mathrm{def}}{=}T(U)/I$, not the space of symetric tensors.

Supplement: local finiteness

E. Let V be a (possibly infinite-dimensional) vector space, $A\in\mathrm{End}(V)$.

(a) Show that the following are equivalent for $\underline{v}\in V$:

(1) $\dim_F\mathrm{Span}_F\{A^n\underline{v}\}_{n=0}^{\infty}<\infty$;

(2) there is a finite-dimensional subspace $\underline{v}\in W\subset V$ such that $AW\subset W$.

DEF Call such $\underline{v}$ *locally finite*, and let V_{fin} be the set of locally finite vectors.

(b) Show that V_{fin} is a subspace of V.

(c) Call A *locally nilpotent* if for every $\underline{v}\in V$ there is $n\geq 0$ such that $A^n\underline{v}=\underline{0}$ (condition (1) of 5(a)). Find a vector space V and a locally nilpotent map $A\in\mathrm{End}(V)$ which is not nilpotent.

(*d) A is called *locally finite* if $V_{\mathrm{fin}}=V$, that is if every vector is contained in a finite-dimensional A-invariant subspace. Find a space V and locally finite linear maps $A,B\in\mathrm{End}(V)$ such that $A+B$ is not locally finite.