Math 100C – SOLUTIONS TO WORKSHEET 8
DIFFERENTIAL EQUATIONS

1. Manipulating Taylor expansions

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

In addition we have the following expansions about $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ; \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

(1) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

Solution: Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f^{(2)}(x) = -\sin x$ and $f^{(3)}(x) = -\cos x$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$ and the third-order expansion of $\sin x$ is $0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 = x - \frac{1}{6}x^3$. In particular $\sin 0.01 \approx 0.01 - \frac{1}{6 \cdot 10^6}$. We then also have, correct to third order, that

$$(x+1)\sin x \approx (x+1)\left(x - \frac{1}{6}x^3\right) = x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 \approx x + x^2 - \frac{1}{6}x^3 .$$

(2) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

Solution: Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, $f^{(2)}(x) = -\frac{1}{4x^{3/2}}$ and $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$. Thus $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f^{(2)}(4) = -\frac{1}{32}$, $f^{(3)}(4) = \frac{3}{256}$ and the third-order expansions are

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{32 \cdot 2!}(x-4)^3 + \frac{3}{256 \cdot 3!}(x-4)^3$$
$$\frac{1}{4}x \approx 1 + \frac{1}{4}(x-4)$$

so that

$$\sqrt{x} - \frac{1}{4}x \approx 1 - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 .$$

(3) Expand $\frac{e^{x^2}}{1+x}$ to second order about $x = 1$.

Solution: Let $x = 1 + h$ so that we are thinking of h as a small variable. We then have $\frac{e^{x^2}}{1+x} = \frac{e^{1+2h+h^2}}{2+h} = \frac{e}{2} \cdot \frac{e^{2h+h^2}}{1+\frac{h}{2}}$ where $2h + h^2$ and $\frac{h}{2}$ are small. Now to second order we have $e^u \approx 1 + u + \frac{u^2}{2}$ and $\frac{1}{1-v} \approx 1 + v + v^2$. Plugging in $u = 2h + h^2$ and $v = -\frac{h}{2}$ we get

$$\begin{aligned} e^{2h+h^2} &\approx 1 + \left(2h + h^2\right) + \frac{1}{2}\left(2h + h^2\right)^2 \\ &= 1 + 2h + h^2 + \frac{1}{2}(4h^2 + 4h^3 + h^4) \\ &\approx 1 + 2h + 3h^2 \end{aligned}$$

and

$$\frac{1}{1 + \frac{h}{2}} \approx 1 + \left(-\frac{h}{2}\right) + \left(-\frac{h}{2}\right)^2 = z1 - \frac{1}{2}h + \frac{1}{4}h^2 ,$$

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correct to second order. We thus have

$$\frac{e^{2h+h^2}}{1+\frac{h}{2}} \approx \left(1+2h+3h^2\right)\left(1-\frac{1}{2}h+\frac{1}{4}h^2\right)$$
$$= 1+\left(1\cdot(-\frac{1}{2})+2\cdot 1\right)h+\left(1\cdot\frac{1}{4}+2\cdot(-\frac{1}{2})+3\cdot 1\right)h^2+\text{higher order}$$
$$\approx 1+\frac{3}{2}h+\frac{9}{4}h^2$$

and (recalling that $h = x-1$)

$$\frac{e^{x^2}}{1+x} = \frac{e}{2}\cdot\frac{e^{2h+h^2}}{1+\frac{h}{2}}$$
$$\approx \frac{e}{2}\left(1+\frac{3}{2}h+\frac{9}{4}h^2\right)$$
$$= \frac{3}{2}+\frac{3e}{4}(x-1)+\frac{9e}{8}(x-1)^2 .$$

(4) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

Solution: To fourth order we have $e^u \approx 1+u+\frac{u^2}{2}+\frac{u^3}{6}+\frac{u^4}{24}+\frac{u^5}{120}$ so $e^{x^2} \approx 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\frac{x^8}{24}$ to 8th order. We also know that $\frac{1}{1-u} \approx 1+u+u^2+u^3$ so $\frac{1}{1+x^3} \approx 1-x^3+x^6$ correct to 8th order. We conclude that

$$e^{x^2}+\cos(2x) \approx \left(1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\frac{x^8}{24}\right)-\left(1-x^3+x^6\right)$$
$$\approx x^2-x^3+\frac{1}{2}x^4-\frac{5}{6}x^6+\frac{1}{24}x^8 .$$

In particular, $\frac{f^{(6)}(0)}{6!} = -\frac{5}{6}$ so $f^{(6)}(0) = -720\cdot\frac{5}{6} = -600$.

(5) Show that $\log\frac{1+x}{1-x} \approx 2(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.

Solution: Let $f(x) = \log(1+x)$. Then $f'(x) = \frac{1}{1+x}$, $f^{(2)}(x) = -\frac{1}{(1+x)^2}$, $f^{(3)}(x) = \frac{1\cdot 2}{(1+x)^3}$, $f^{(4)}(x) = -\frac{1\cdot 2\cdot 3}{(1+x)^4}$ and so on, so $f^{(k)}(x) = (-1)^{k-1}\cdot\frac{(k-1)!}{(1+x)^k}$. We thus have that $f(0) = 0$ and for $k \geq 1$ that $f^{(k)}(0) = (-1)^{k-1}(k-1)!$. Then $\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k-1}}{k}$ so

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots .$$

Plugging $-x$ we get:

$$\log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}\cdots$$

so

$$\log\frac{1+x}{1-x} = \log(1+x)-\log(1-x) = 2x+2\frac{x^3}{3}+2\frac{x^5}{5}+\cdots .$$

In particular

$$\log 3 = \log\frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 2\left(\frac{1}{2}+\frac{1}{24}+\frac{1}{160}+\cdots\right) = 1+\frac{1}{12}+\frac{1}{80}+\cdots \approx 1.096$$

2. Differential equations

(6) For each equation: Is $y = 3$ a solution? Is $y = 2$ a solution? What are *all* the solutions?

$$y^2 = 4 \qquad ; \qquad y^2 = 3y$$

Solution: Plugging in 2 we have $2^2 = 4$ in the first equation but $2^2 \neq 3\cdot 2$. Plugging in 3 we have $3^2 \neq 4$ but $3^2 = 3\cdot 3$. The solutions to the first equations are $\{\pm 2\}$, to the second $\{0, 3\}$.

(7) For each equation: Is $y(x) = x^2$ a solution? Is $y(x) = e^x$ a solution?

$$\frac{dy}{dx} = y \qquad ; \qquad \left(\frac{dy}{dx}\right)^2 = 4y$$

Solution: Plugging in $y = x^2$ into the equations we have $2x \neq x^2$ but $(2x)^2 = 2 \cdot x^2$ is true. Plugging in e^x into the equations we see $e^x = e^x$ but $(e^x)^2 = e^{2x} \neq 4e^x$.

(8) Which of the following (if any) is a solution of $\frac{dz}{dt} + t^2 - 1 = z$ (challenge: find more solutions):

A. $z(t) = t^2$; B. $z(t) = t^2 + 2t + 1$

Solution: $2t + t^2 - 1 \neq t^2$ but $(2t+2) + t^2 - 1 = t^2 + 2t + 1$ so only B is a solution. If w is another solution them we have

$$\frac{dw}{dt} + t^2 - 1 = w$$
$$\frac{dz}{dt} + t^2 - 1 = z$$

and subtracting the two equations we get $\frac{d(w-z)}{dt} = w - z$ so $w - z = Ce^t$ and $w(t) = Ce^t + t^2 + 2t + 1$ for any constant t.

(9) The balance of a bank account satisfies the differential equation $\frac{dy}{dt} = 1.04y$ (this represents interest of 4% compounded continuously). Sketch the solutions to the differential equation. What is the solution for which $y(0) = \$100$?

Solution: The solutions are $Ce^{1.04t}$ for arbitrary C. The particular solution is $100e^{1.04t}$ dollars.

(10) Suppose $\frac{dy}{dx} = ay$, $\frac{dz}{dx} = bz$. Can you find a differential equation satisfied by $w = \frac{y}{z}$? Hint: calculate $\frac{dw}{dx}$.

Solution: $w' = \left(\frac{y}{z}\right)' = \frac{y'z - yz'}{z^2} = \frac{ayz - ybz}{z^2} = (a-b)\frac{y}{z} = (a-b)w$ so the equation is $\frac{dw}{dx} = (a-b)w$.

3. Solutions by massaging and ansatze

(11) For which value of the constant ω is $y(t) = \sin(\omega t)$ a solution of the oscillation equation $\frac{d^2y}{dt^2} + 4y = 0$?

Solution: $(\sin(\omega t))' = \omega \cos \omega t$ so $(\sin(\omega t))'' = -\omega^2 \sin(\omega t)$ so

$$(\sin(\omega t))'' = -4\left(\sin(\omega t)\right)$$

iff $\omega^2 = 4$, that is iff $\omega = \pm 2$.

(12) (The quantum harmonic oscillator) For which value of the constants A, B (with $B > 0$) does the function $f(x) = Axe^{-Bx^2}$ satisfy $-f'' + x^2 f = 3f$? What if we also insist that $f(1) = 1$?

Solution: $f' = Ae^{-Bx^2} - 2ABx^2e^{-Bx^2}$ so $f'' = -6ABxe^{-Bx^2} + 4AB^2x^3e^{-Bx^2}$ and

$$-f'' + x^2 f = 6ABxe^{-Bx^2} + \left(Ax^3e^{-Bx^2} - 4AB^2x^3e^{-Bx^2}\right)$$
$$= 6ABxe^{-Bx^2} + A\left(1 - 4B^2\right)x^3e^{-Bx^2}$$

so

$$-f'' + x^2 f = \left(6B + (1 - 4B^2)x^2\right)Axe^{-Bx^2}$$

and we get a solution to our equation only if $1 - 4B^2 = 0$ that is if $B = \frac{1}{2}$ (and then $6B = 3$ as desired). Finally the solution has $f'(1) = 1$ if $Ae^{-1/2} = 1$ so $A = e^{1/2}$ and $f(x) = xe^{-\frac{1}{2}(x^2-1)}$.

(13) Consider the equation $\frac{dy}{dt} = a(y - b)$.

(a) Define a new function $u(t) = y(t) - b$. What is the differential equation satisfied by uv?
Solution: $u' = y' = a(y - b)' = au$.

(b) What is the general solution for $u(t)$?
Solution: $u(t) = Ce^{at}$ where $C = u(0)$.

(c) What is the general solution for $y(t)$?
Solution: $y(t) = u(t) + b = Ce^{at} + b$.

(d) Suppose $a < 0$. What is the asymptotic behaviour of the solution as $t \to \infty$?
Solution: $y(t) \xrightarrow[x\to\infty]{} b$ and the convergence is exponential: $y(t) - b$ decays exponentially.

(e) Suppose we are given the *initial value* $y(0)$. What is C? What is the formula for $y(t)$ using this?

Solution: We have $Ce^{a\cdot 0} + b = y(0)$ so $C = y(0) - b$ and $y(t) = (y(0) - b)\, e^{at} + b\,.$