6. Curve sketching; Taylor expansion (20/10/2022)

Goals.

(1) Clarify inverse trig
(2) Curve sketching
(3) Taylor expansion

Last Time. (1) chain rule: $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

$\Rightarrow$ (2) implicit diff: evaluate $\frac{dF}{dx}(x,y)$ along curve $y=f(x)$. (e.g. if $x^3+y^2=7x$ along curve, chain rule then $3x^2+2y\cdot y'=7$ along the curve)

$\Rightarrow$ (3) log diff: $\frac{dy}{dx} = y \cdot \frac{d(\log y)}{dx}$

idea: maybe $\frac{d(\log y)}{dx}$ is easier

(4) Inverse trig functions:

Answer: "what angle θ has sin/cos/tan equal to x?"

Domain = range of sin/cos/tan

range: $\arcsin x$ def if $x \in [-1,1]$, takes values in $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arccos x$ " " " , " " in $\theta \in [0, \pi]$

$\arctan x$ " for all $x \in \mathbb{R}$, " " $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Derivatives:

If $\theta = \arcsin x$ then $x = \sin\theta$, diff wrt x

get $1 = \cos\theta \cdot \frac{d\theta}{dx}$ so $\frac{d\theta}{dx} = \frac{d(\arcsin x)}{dx} = \frac{1}{\cos\theta}$

Pythagoras: $\sin^2\theta + \cos^2\theta = 1$ $\rightarrow = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$

Similary: $\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\frac{d(\arccos x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$

E.g. $\theta = \operatorname{arcsec} x \Rightarrow \sec\theta = x \Rightarrow \frac{1}{\cos\theta} = x$

$\Rightarrow \cos\theta = \frac{1}{x} \Rightarrow \theta = \arccos\frac{1}{x}$

Math 100C – WORKSHEET 6
CURVE SKETCHING; TAYLOR EXPANSION

1. Curve sketching

Let $f(x) = \frac{x^3+2}{x^2+1}$; ~~and that $f''(x) = -2\frac{x^3-6x^2-3x+2}{(x^2+1)^3}$~~

(1) Zeroeth derivative questions

(a) Where is f defined?

Everywhere: since $x^2+1 \geq 1 > 0$ for all x, $\frac{1}{x^2+1}$ is always defined

(b) List the vertical asymptotes of f, if any?

f is cts everywhere (def by formula) so does not blow up, has no vertical asymptotes

(c) What are the asymptotic behaviours of f at $\pm\infty$?

If $|x|$ is big, $\frac{x^3+2}{x^2+1} \sim \frac{x^3}{x^2} = x$ so as $x \to \infty$ or $-\infty$, $f(x) \sim x$.

(d) Where does f meet the axes?

$f(0) = 2$, $f(x) = 0$ if $x^3+2=0$ so $x = -\sqrt[3]{2}$.
(actually $f(x) < 0$ if $x < -\sqrt[3]{2}$, $f(x) > 0$ if $x > -\sqrt[3]{2}$)

Date: 20/10/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) It is a fact that $f'(x) = \frac{x(x-1)(x^2+x+4)}{(x^2+1)^2}$

(a) Where is f differentiable?

Everywhere, since $x^2+1>0$ again.

(b) Where does $f'(x) = 0$? Where it is positive? Negative?

$f'(0) = f'(1) = 0$, f' nonzero otherwise. $x^2+x+4 = (x+\frac{1}{2})^2 + \frac{15}{4} > 0$

~~The fact~~ $\Rightarrow$ $f'(x) = x(x-1)\left[\begin{array}{c}\text{positive}\\ \text{quantity}\end{array}\right]$

Positive if $x>1$ or $x<0$, negative if $0<x<1$

(c) Where are the local extrema of f? What are the values at those points?

$x=0$ is a local maximum, $x=1$ is a local min

$f(0) = 2$, $\qquad f(1) = \frac{3}{2}$

(3) It is a fact that $f''(x) = -2\frac{x^3-6x^2-3x+2}{(x^2+1)^3}$.

(a) Where is f'' positive/negative? Where does it vanish? Say as much as you can.

(b) Where is f concave up/down? Where are its inflection points?

$$f''(x) = -\frac{2}{\underbrace{(x^2+1)^3}_{\text{positive}}}(x^3-6x^2-3x+2)$$

if $x \to \pm\infty$ $\quad x^3-6x^2-3x+2 \sim x^3$

so $f''(x)$ is >0 if x very negative

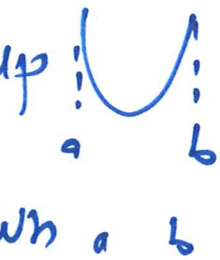
$f''(x)$ is <0 if x " positive

$f''(0) = \frac{-4}{1} < 0$, $\quad f''(-1) = \frac{+10}{2^3} > 0$

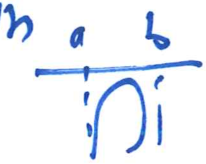
$f''(1) = \frac{12}{2^3} > 0$

so 2^{nd} derivative starts >0, . . .

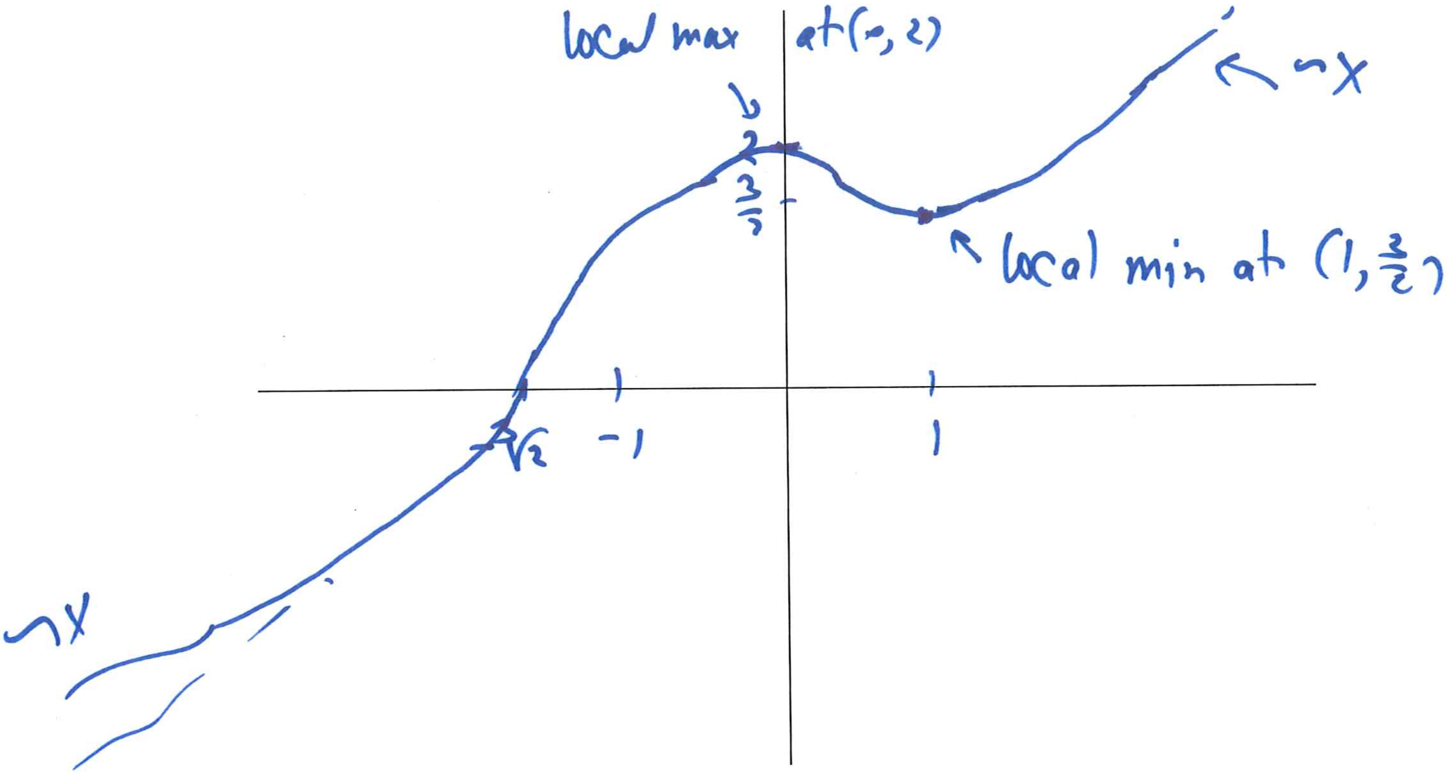
If $f''(x) > 0$ on $[a,b]$, f is concave up



If $f''(x) < 0$ on (a,b), f is concave down



(4) Draw a sketch of the graph of f, incorporating all the features you have identified in questions 1-3.



- Extra credit: Find the constant b so that $f(x) \approx x+b$ as $x \to \infty$ (in the sense that $f(x)-x-b \to 0$). We call this line a *slant asymptote* for f.

2. Taylor expansion

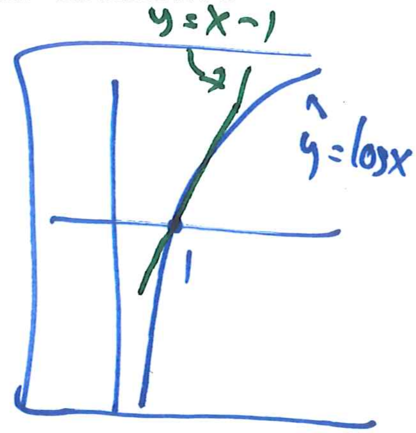
(5) (Review) Use linear approximations to estimate:

(a) $\log \frac{4}{3}$ and $\log \frac{2}{3}$. Combine the two for an estimate of $\log 2$.

Let $a = 1$, $f(x) = \log x$.

$f(a) = \log 1 = 0$

$f'(a) = \frac{1}{1} = 1$



so $f(x) = \log x \approx 0 + 1\cdot(x-1) = x-1$

so $\log\left(\frac{4}{3}\right) \approx \frac{1}{3}$

$\log\left(\frac{2}{3}\right) \approx -\frac{1}{3}$

$\left(\log 2 = \log \frac{4/3}{2/3} = \log \frac{4}{3} - \log \frac{2}{3} \approx 2/3\right)$

(b) $\sin 0.1$ and $\cos 0.1$.

(two formulas: $f(a+h) \approx f(a) + f'(a)h$

$f(x) \approx f(a) + f'(a)(x-a)$)

$\sin 0 = 0$, $(\sin\theta)'\big|_{\theta=0} = \cos 0 = 1 \rightarrow \sin 0.1 \approx 0 + 1\cdot 0.1 = 0.1$

$\cos 0 = 1$, $(\cos\theta)'\big|_{\theta=0} = \sin 0 = 0 \rightarrow \cos 0.1 \approx 1 + 0\cdot 0.1 = 1$

(6) Let $f(x) = e^x$

(a) Find $f(0), f'(0), f^{(2)}(0), \cdots$

(b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

(c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

(d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

(e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(a) $f'(x) = e^x$, $f''(x) = e^x$, $f^{(3)}(x) = e^x$, ...

so $f^{(n)}(0) = 1$ for all n.

$$\left[f^{(0)}(x) = f(x),\quad f^{(1)}(x) = f'(x),\quad f^{(n+1)}(x) = \frac{d f^{(n)}(x)}{dx} \right]$$

(b) Take $T_0(x) = 1$ (if x close to 0, e^x close to 1 by continuity)

(c) Take $T_1(x) = \underbrace{1}_{\text{value}} + \underbrace{1}_{\text{slope}} \cdot x$ (linear approx)

(d) try: $p(x) = 1 + x^2$ $\quad p(0) = 1$ ✓ $\quad p'(x) = 2x \quad p'(0) = 0$

try: $p(x) = 1 + x + x^2$ $\quad p(0) = 1$ ✓ $\quad p'(x) \overset{!}{=} 1 + 2x \quad p'(0) = 1$ ✓

try: $p(x) = 1 + x + \frac{x^2}{2}$ $\quad p(0) = 1$, $p'(x) = 1 + x$, $p'(0) = 1$, $p''(x) = 2x$, $p''(x) = 1$ ✓

(7) Do the same with $f(x) = \log x$ about $x = 1$.

For 3^{rd} order, try $1+x+\frac{x^2}{2}+cx^3$

diff 3 times get: $\left(1+x+\frac{x^2}{2}+cx^3\right)^{(3)} = c\cdot 3\cdot 2\cdot 1$

so to get 1 need $c = \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6}$

Conclusion: e^x near $a=0$,

$e^x \approx 1$ to 0^{th} order

$e^x \approx 1+x$ to 1^{st} order

$e^x \approx 1+x+\frac{x^2}{2}$ to 2^{nd} order

$e^x \approx 1+x+\frac{x^2}{2}+\frac{x^3}{6}$ to 3^{rd} order

Lesson: Can approximate f near a by a polynomial by matching its derivatives

Formula: polynomial is $C_0 + C_1(x-a) + C_2(x-a)^2 + \ldots$

k'th coeff is $\frac{f^{(k)}(a)}{k!}$ ← $1\cdot 2\cdot 3 \cdots k$

$C_0 = \frac{f(a)}{1}$, $C_1 = \frac{f'(a)}{1}$, $C_2 = \frac{1}{2} f^{(2)}(a)$, $C_3 = \frac{1}{6} f^{(3)}(a)$

↖ $1\cdot 2$ ↖ $1\cdot 2\cdot 3$

$C_4 = \frac{1}{24} f^{(4)}(0)$, ..

↑ $1\cdot 2\cdot 3\cdot 4$

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(8) Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

Let $g(x) = \frac{1}{1-x}$, $g'(x) = -1 \cdot (1-x)^{-2} \cdot (-1) = (1-x)^{-2}$

$= (1-x)^{-1}$

$g''(x) = -2(1-x)^{-3}(-1) = 2 \cdot (1-x)^{-3}$

$g^{(3)}(x) = 2 \cdot 3 (1-x)^{-4}$, $g^{(4)}(x) = 2 \cdot 3 \cdot 4 (1-x)^{-5}$

$g(0) = 1$, $g'(0) = 1$, $g^{(2)}(0) = 2$, $g^{(3)}(0) = 2 \cdot 3$, $g^{(4)}(0) = 2 \cdot 3 \cdot 4$

So $T_4(x) = 1 + \frac{1}{\underset{1!}{1}} x + \frac{2}{\underset{2!}{1 \cdot 2}} x^2 + \frac{2 \cdot 3}{\underset{3!}{1 \cdot 2 \cdot 3}} x^3 + \frac{2 \cdot 3 \cdot 4}{\underset{4!}{1 \cdot 2 \cdot 3 \cdot 4}} x^4$

$= 1 + x + x^2 + x^3 + x^4$

$\frac{1}{1-5y} \neq 1 + 5y$

$\approx 1 + \frac{1}{20} + (\frac{1}{20})^2$

(9) Find the nth order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

If $f(x) = \cos x \quad f'(x) = -\sin x \quad f''(x) = -\cos x$

$f^{(3)}(x) = +\sin x$

$f^{(4)}(x) = \cos x, \ldots$ repeat

so $f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \ldots$ repeat

$\Rightarrow \cos x \simeq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \ldots$

$\Rightarrow \cos 0.1 \simeq 1 - \frac{1}{200}$ to 3^{rd} order

(10) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

Sol $\quad 12 = \dfrac{f''(3)}{2!}$ so $f''(3) = 24$

(can use formula $c_k = \dfrac{f^{(k)}(a)}{k!}$ in reverse)