Math 100 – SOLUTIONS TO WORKSHEET 18
THE MVT AND CURVE SKETCHING

1. The shape of a the graph

(1) Side exercise: Let f be twice differentiable on $[a,b]$.

(a) Suppose first that $f(a)=f(b)=0$ and that f is positive somewhere between a,b. Show that there is c between a,b so that $f''(c)<0$.
Solution: Suppose $a<x<b$ so that $f(x)>0$. By the MVT there are $y\in(a,x)$ and $z\in(x,b)$ so that

$$f'(y)=\frac{f(x)-f(a)}{x-a}=\frac{f(x)}{x-a}>0$$
$$f'(z)=\frac{f(b)-f(x)}{b-x}=-\frac{f(x)}{b-a}<0\,.$$

In particular, $f'(y)>f'(z)$ but $y<z$ so $f'(z)-f'(y)<0$ but $z-y>0$. Applying the MVT to the twice differentiable function f' on the interval $[y,z]$ gives $c\in[y,z]\subset(a,b)$ such that

$$f''(c)=\frac{f'(z)-f'(y)}{z-y}<0\,.$$

(b) Now let $f(a),f(b)$ take any values, but suppose $f''(x)>0$ on (a,b). Let $L:y=mx+n$ be the line through $(a,f(a)),(b,f(b))$. Applying part (a) to $g(x)=f(x)-(mx+n)$ show that the graph of f lies below the line L.
Solution: Let $g(x)=f(x)-(mx+n)$. Since the line passes through $(a,f(a))$ and $(b,f(b))$ we have $g(a)=g(b)=0$. Also, for all $a<x<b$, $g''(x)=f''(x)>0$ since $(mx+n)''=0$. It follows that there is no point such that $g(x)>0$, so $g(x)\le 0$ that is $f(x)\le mx+n$.

(2) For each of the following functions determine its domain, and where it is increasing or decreasing. Except in part (b) also determine where the function is concave up/down.

(a) $f(x)=e^x$
Solution: f is defined everywhere. $f'(x)=e^x$ which is everywhere positive, so f is everywhere increasing. Similarly $f''(x)=e^x$ is everywhere positive, so f is concave up on the whole line.

(b) $f(x)=\frac{x-1}{x^2+1}$
Solution: $f'(x)=\frac{(x^2+1)-(x-1)(2x)}{(1+x^2)^2}=\frac{1+2x-x^2}{(1+x^2)^2}=-\frac{(x-1)^2-2}{(1+x^2)^2}$ which is positive when $x>1+\sqrt{2}$ and $x<1-\sqrt{2}$ and negative on $\left(1-\sqrt{2},1+\sqrt{2}\right)$, so the function is increasing on $\left(-\infty,1-\sqrt{2}\right)$, decreasing on $\left(1-\sqrt{2},1+\sqrt{2}\right)$ and then increasing on $\left(1+\sqrt{2},\infty\right)$.

(c) $f(x)=x\log x-2x$
Solution: $f'(x)=\log x+1-2=\log x-1$, which is positive when $x>e$ and negative when $0<x<e$ (the function is undefined if $x<0$). Thus the function is increasing on $(0,e)$ and decreasing on (e,∞). We have $f''(x)=\frac{1}{x}$ which is positive on the entire domain so the function is concave up everywhere.

(d) $\frac{x^2-9}{x^2+3}$. You may use that $f'(x)=\frac{24x}{(x^2+3)^2}$ and that $f''(x)=72\frac{1-x^2}{(x^2+3)^3}$.
Solution: f is defined everywhere. The derivative is positive for $x>0$, negative for $x<0$. The second derivative is positive for $-1<x<1$ and negative otherwise. The function is therefore decreasing for $x<0$ and increasing for $x>0$ (minimum at $x=0$!), concave down in $(-\infty,-1)$ and $(1,\infty)$ and concave up on $(-1,1)$ with inflection points at ± 1.
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