Math 100 – WORKSHEET 12
EXPONENTIAL GROWTH AND DECAY

1. More Related Rates

(1) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.
   (a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

   (b) The drain is unclogged and water begins to drain at the rate of $(5+\frac{\pi}{4})\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

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(2) Two ships are travelling near an island. The first is located 20km due west of it, The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing if:

(a) The first ship is moving due north at 5km/h.

(b) The same setting, but now the first ship is moving toward the island.

2. Exponential growth and decay

Growth/decay described by the *differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\,,$$

Solution: $y =$

(3) Suppose[1] that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.
   (a) At what time will there be 1000 opossums in BC? 10,000 opossums?

   (b) Write a differential equation expressing the growth of the opossum population with time.

(4) A radioactive sample decays according to the law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km\,.$$

   (a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?
   (b) A 100-gram sample is left unattended for three days. How much of it remains?

(5) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.

[1] See http://linnet.geog.ubc.ca/efauna/Atlas/Atlas.aspx?sciname=Didelphis%20virginiana

3. Newton's Law of Cooling

Fact. *When a body of temperature T_0 is placed in an environment of temperature T_{env} the temperature difference $T(t) - T_{env}$ between the body and the environment decays exponentially. In other words, there is a (negative) constant k such that*

$$T' = k(T - T_{env}) \qquad\qquad T(t) - T_{env} = (T_0 - T_{env})e^{kt}\,.$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\mathrm{env}}$. Then
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = ky$$

Corollary. $\lim_{t\to\infty} y(t) = 0$ *so* $\lim_{t\to\infty} T(y) = T_{env}$.

(6) (Final, 2010) When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.
   (a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.

   (b) Determine the time when the temperature of the apple is $16°C$.

   (c) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.

(7) (Final, 2013) A bottle of soda pop at room temperature ($70°F$) is placed in the refrigerator where the temperature is $40°F$. After half han hour the bottle has cooled to $60°F$. When will it reach $50°F$?