Math 100, Lecture 23, 2/12/2021

Review 2

form 1^∞

Q: Evaluate $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x/4} \underset{y=x/2}{=} \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y/2} \underset{\text{continuity of } (\cdot)^{1/2}}{=} \left(\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right)^{1/2}$

Know: $\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}=e$, answer is $e^{1/2}=\sqrt{e}$

If don't know, take logs: $\log\left(1+\frac{1}{y}\right)^{y} = y\log\left(1+\frac{1}{y}\right)$

$\infty\cdot 0$

Warning: an indeterminate forms can have any limit!

compare

$$\frac{1}{x}\cdot x^2 \xrightarrow[x\to\infty]{} \infty$$

$$\frac{1}{x}\cdot x \xrightarrow[x\to\infty]{} 1$$

$$\frac{1}{x^2}\cdot x \xrightarrow[x\to\infty]{} 0$$

all of the form $0\cdot\infty$

Optimization / Related Rates notes

(0) Read problem: understand the idea, draw a picture if possible.
(1) Assign names:
  - Choose axes, quantities of interest.
  - Give a *name* to each quantity of interest.
(2) Function/relations: express quantity to be optimized as a function of the dependent variable.
  - Sometimes the quantity depends on several variables, and we need to enforce *relations* between them to end up with one independent variable.
(3) Calculus: find domain and the minima and maxima on the domain.
  - (Related rates: use the chain rule when differentiating).
(4) Interpretation: solve the problem using the calculus result.
  - Make *sanity checks* (area can't be negative, for example).

Q: (Related rates)

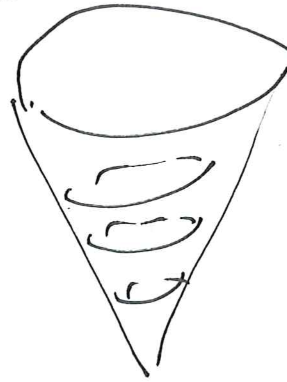
A tank of water in the shape of an inverted cone is leaking water at a constant rate. The height is 10 ft, the radius of the base is 5 ft.

(1) At what rate is the depth of the water changing when the depth is 6 ft? (leak at rate $2\ ft^3/hr$)

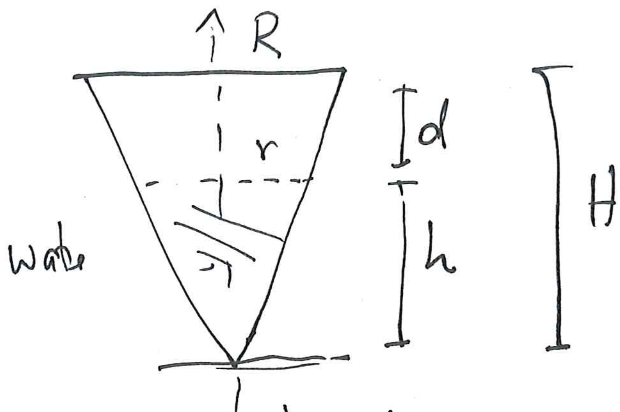
(2) At what rate is the radius at top changing?

diagram

Cone



vertical cross-section



names

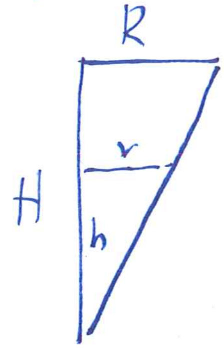
let R = radius of cone, H = height

r = radius of water, h = height of water.

d = depth of water. $V = \frac{1}{3}\pi r^2 h$ = volume of water.

water takes shape of a cone, of volume ↑

look at the triangles



similarity of triangles gives

$$\frac{r}{h} = \frac{R}{H}$$

so $r = \frac{R}{H} \cdot h$,

$$V = \left(\frac{1}{3}\pi \frac{R^2}{H^2}\right) h^3$$

relations

And $h + d = H$

So $V = \frac{1}{3}\pi \frac{R^2}{H^2}(H-d)^3$

$$\frac{dV}{dt} = \frac{1}{3}\pi \frac{R^2}{H^2}(H-d)^2 \cdot 3 \cdot \left(-\frac{dd}{dt}\right)$$ ← Calculus

endgame

Here, $\frac{dV}{dt} = -2 \frac{ft^3}{hr}$ $\quad R = 5\text{ ft},\ H = 14\text{ ft},$

$d = 6\text{ ft}$

So $-2 = \pi \cdot \frac{25}{196} \cdot 64 \cdot -\frac{dd}{dt}$

So $\boxed{\frac{dd}{dt} = \frac{49}{200\pi} \frac{ft}{hr}}$ when depth is 6ft, it is increasing at the rate $\frac{49}{200\pi}$ ft/hr

earlier relation

(2) $r = \frac{R}{H} \cdot h$

$$\frac{dr}{dt} = \frac{R}{H}\frac{dh}{dt} = \frac{R}{H} \cdot -\frac{dd}{dt} = -\frac{5}{14} \cdot \frac{49}{200\pi} \text{ ft/hr}$$

more calc

$h + d = H$ $\quad$ when $d = 6$ ft

$$\boxed{= -\frac{7}{80\pi} \text{ ft/hr}}$$

i.e. the radius is decreasing at the rate of $\frac{7}{80\pi}$ ft/hr.

more endgame

Q: (Lagrange Remainder)

Suppose we approximate $f(x)$ by $T_n(x)$ where T_n is the Taylor polynomial of f about $x = a$.

How can we tell if $T_n(x) > f(x)$ ("overestimate")

or $T_n(x) < f(x)$ ("underestimate")?

A: $f(x) = T_n(x) + R_n(x) = T_n(x) + \frac{1}{(n+1)!} f^{(n+1)}(c) (x-a)^{n+1}$

with c between a, x.

See: $T_n(x) < f(x) \iff R_n(x) > 0$

$T_n(x) > f(x) \iff R_n(x) < 0$

We know the sign of $(x-a)^{n+1}$

If we also know the sign of $f^{(n+1)}(c)$, we know the sign of $R_n(x)$

we don't know c! By maybe $f^{(n+1)}$ does not change sign on $[a, x]$

Q: Let $f(x) = (1 + \frac{x}{2})^{1/3}$, find the first three terms of Taylor expansion about 0.

$f(0) = 1$, $f'(0) = \frac{1}{6}$, $f''(0) = -\frac{1}{18}$

$f'(x) = \frac{1}{3}(1+\frac{x}{2})^{-2/3} \cdot \frac{1}{2}$

$f''(x) = -\frac{1}{18}(1+\frac{x}{2})^{-5/3}$

So

$T_2(x) = \underset{f(0)}{1} + \underset{f'(0)}{\frac{1}{6}} x - \underset{\frac{1}{2!} f^{(2)}(0)}{\frac{1}{36}} x^2$

Q: (Taylor expansion)

Determine the Taylor expansion of $f(x) = \sin x \cdot \cos x$ about $x = \pi$

note: $f(x) = \frac{1}{2} \cdot 2\sin x \cos x = \frac{1}{2}\sin(2x)$ $\qquad f(\pi) = 0$

① $\qquad f'(\pi) = 1$

$f'(x) = \cos(2x)$

$f''(x) = -2\sin(2x)$ $\qquad f''(\pi) = 0$

$f^{(3)}(x) = -4\cos(2x)$ $\qquad f^{(3)}(\pi) = -4$

$f^{(4)}(x) = 8\sin 2x$ $\qquad f^{(4)}(\pi) = 0$

So $T_4(x) = (x-\pi) - \frac{4}{3!}(x-\pi)^3 = \boxed{(x-\pi) - \frac{2}{3}(x-\pi)^3}$

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② $\sin \pi = 0$ $\qquad \cos$

$(\sin')(\pi) = \cos\pi = -1$ $\qquad \cos \pi = -1$

$(\sin'')(\pi) = -\sin\pi = 0$ $\qquad (\cos')(\pi) = 0$

$(\sin''')(\pi) = -\cos\pi = 1$ $\qquad 1$

$(\sin^{(4)})(\pi) = 0$ $\qquad 0$

$\qquad -1$

So to 4th order, $\sin x \approx -(x-\pi) + \frac{1}{6}(x-\pi)^3$

$\cos x \approx -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4$

So $\sin x \cos x \approx \left(-(x-\pi) + \frac{1}{6}(x-\pi)^3\right)\left(-1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4\right)$

ignored terms of deg > 4 $\rightarrow$ $\approx (x-\pi) - \frac{1}{6}(x-\pi)^3 - \frac{1}{2}(x-\pi)^3 \approx \boxed{(x-\pi) - \frac{2}{3}(x-\pi)^3}$

Recap: can expand complicated function to order n by expanding each piece separately, combining the expansions, discarding terms of order $> n$

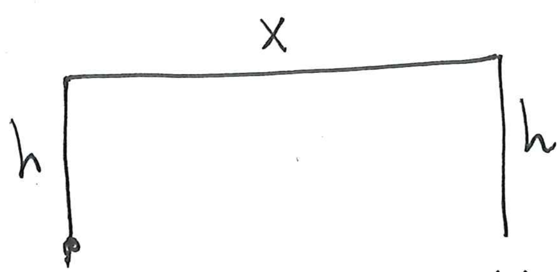
(could also have differentiated $\sin x \cdot \cos x$ repeatedly)

Q: (Optimization)

Have 500ft of fencing, want to enclose largest possible rectangular area on 3 sides.

← diagram

A:



names

say the rectangle has width x, height h

say the fencing has length L, rectangle has area A

then $L = x + 2h$. $\quad A = xh$ $\quad$ relations

so $h = \dfrac{L-x}{2}$ $\quad$ so $A = x \cdot \dfrac{L-x}{2}$

makes sense if $0 \le x \le L$

Need to maximize $A(x) = x \cdot \frac{L-x}{2}$ on $[0, L]$

$A'(x) = \frac{L-x}{2} - \frac{1}{2}x = \frac{L}{2} - x \Rightarrow$ unique critical pt at $\frac{L}{2}$.

Calculus

Now $A(0) = A(L) = 0$, $A(\frac{L}{2}) = \frac{1}{8}L^2$

area (sanity check)

$\boxed{\text{So max is if } x = \frac{L}{2},\ h = \frac{L}{4}}$

max area is $\frac{1}{8}L^2$.

The largest field is 250 ft × 125 ft.

Endgame