20. L'HOPITAL'S RULE (23/11/2021)

Goals.

(1) Compute limits using l'Hopital's rule

(2) Connect to Taylor expansion

Last Time. **Curve sketching**

Today: l'Hôpital's rule

If we want to compute $\lim_{x \to a} \frac{f(x)}{g(x)}$ (or $\lim_{x \to \infty}$, $-\infty$)

and either $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or both are infinite (in extended sense)

and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists. Then $\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$

Remark: This theorem has hypotheses and a conclusion

To apply theorem need (1) check all hypotheses

(2) invoke theorem by name

limits of the form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" are called "~~[illegible]~~ indeterminate forms

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Why? $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \Big/ \frac{g(x)-g(a)}{x-a}$

if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

Math 100 – WORKSHEET 20
L'HÔPITAL'S RULE

(1) Evaluate $\lim_{x\to 1} \frac{\log x}{x-1}$.

$\lim_{x\to 1} \log x = \log 1 = 0$
$\lim_{x\to 1} x-1 = 1-1 = 0$ } by l'Hôpital's rule,
$\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1$

Aside: $\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \frac{\log x - \log 1}{x-1} = \left[\frac{d}{dx}\log x\right]_{x=1} = \left[\frac{1}{x}\right]_{x=1} = 1$

Aside: $\lim_{x\to 1} \frac{\log x}{x} = \frac{\log 1}{1} = \frac{0}{1} = 0$ but $\lim_{x\to 1} \frac{1/x}{1} = 1$

(2) (Final, 2014) Evaluate $\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2}$.

$\lim_{x\to 0} (\cos x - e^{x^2}) = \cos 0 - e^{0^2} = 1-1 = 0$, $\lim_{x\to 0} x^2 = 0$. By l'Hôpital's rule,
$\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x}$ (if latter limit exists)

Now $\lim_{x\to 0} (-\sin x - 2xe^{x^2}) = -\sin 0 - 0 = 0$; $\lim_{x\to 0} 2x = 0$, By l'Hôpital's rule again,
$\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x} = \lim_{x\to 0} \frac{-\cos x - 2e^{x^2} - 4x^2e^{x^2}}{2}$
$= \frac{-1-2-0}{2} = -3/2$

Date: 23/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(or: $\lim_{x\to 0} \frac{-\sin x - 2xe^{x^2}}{2x} = \lim_{x\to 0}\left[-\frac{1}{2}\cdot\frac{\sin x}{x} - e^{x^2}\right] = -\frac{1}{2} - 1 = -3/2$

(3) Do (2) using a 2nd-order Taylor expansion.

$\cos 0 = 1$; $(\cos')(0) = -\sin 0 = 0$; $(\cos'')(0) = -\cos 0 = -1$

to 2^{nd} order $\cos x \approx 1 - \frac{1}{2}x^2$

$e^0 = 1$, $(e^u)'_{u=0} = 1$, $(e^u)''_{u=0} = 1$, $e^u \approx 1 + u + \frac{u^2}{2}$

set $u = x^2$

$\Rightarrow e^{x^2} \approx 1 + x^2 + \frac{x^4}{2} \approx 1 + x^2$ to 2^{nd} order.

Now $\frac{\cos x - e^{x^2}}{x^2} = \frac{(\cancel{1} - \frac{x^2}{2} + R_2) - (\cancel{1} + x^2 + R_2)}{x^2} = -\frac{3}{2} + \frac{R_2}{x^2} \xrightarrow[x\to 0]{} -3/2$

(4) (Final, 2015) Evaluate $\lim_{x\to 0} \frac{\log(1+x) - \sin x}{x^2}$.

$\lim_{x\to 0} \log(1+x) - \sin x = \log 1 - \sin 0 = 0$; $\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} \frac{1}{1+x} - \cos x = \frac{1}{1+0} - \cos 0 = 0$; $\lim_{x\to 0} 2x = 0$

So by l'Hôpital's rule, twice, we have

$\lim_{x\to 0} \frac{\log(1+x) - \sin x}{x^2} = \lim_{x\to 0} \frac{\frac{1}{1+x} - \cos x}{2x} = \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2} - \sin x}{2} = -\frac{1}{2}$

(or: to 2^{nd} order $\log 1+x \approx x - \frac{x^2}{2}$, $\sin x \approx x - \frac{x^3}{6} \approx x$

So $\log(1+x) - \sin x = (x - \frac{x^2}{2}) - (x) + R_2 = -\frac{x^2}{2} + R_2$

So $\frac{\log(1+x) - \sin x}{x^2} = -\frac{1}{2} + \frac{R_2}{x^2} \xrightarrow[x\to 0]{} -\frac{1}{2}$ (R_2 is of order x^3)

& f, g continuously diff near 2

(5) Given that $f(2) = 5$, $g(2) = 3$, $f'(2) = 7$ and $g'(2) = 4$ find $\lim_{x\to 3} \frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}$.

$$\lim_{x\to 3}\left[f(2x-4) - g(x-1) - 2\right] = f(2) - g(2) - 2 = 5-3-2 = 0$$

f, g cts at 2 because they are differentiable there

$$\lim_{x\to 3}\left[g(x^2-7) - 3\right] = g(2) - 3 = 0$$

By l'Hôpital's rule

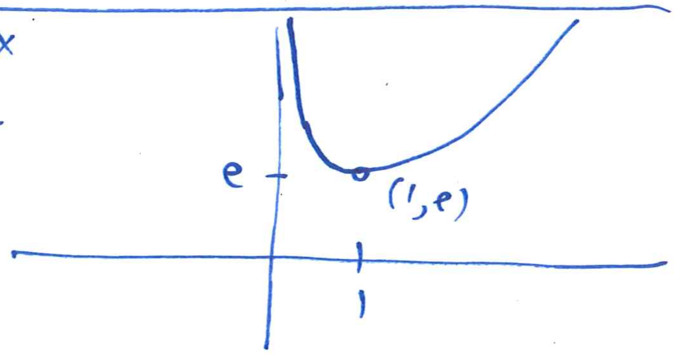
$$\lim_{x\to 3} \frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3} = \lim_{x\to 3} \frac{2f'(2x-4)-g'(x-1)}{g'(x^2-7)\cdot 2x} = \frac{2f'(2)-g'(2)}{g'(2)\cdot 2\cdot 3}$$

(6) Evaluate $\lim_{x\to 0^+} \frac{e^x}{x}$.

$\lim_{x\to 0^+} e^x = 1 \neq 0$; $\lim_{x\to 0^+} x = 0$ so $\frac{e^x}{x}$ blows up at $x=0$,

If $x > 0$ then $\frac{e^x}{x} > 0$ so $\lim_{x\to 0^+} \frac{e^x}{x} = +\infty$

$$\left(\frac{e^x}{x}\right)' = \frac{xe^x - e^x}{x^2} = \frac{(x-1)e^x}{x^2}$$



How to tell $\lim_{x\to\infty} \frac{e^x}{x} = \infty$?

$$\left.\begin{aligned}\lim_{x\to\infty} e^x &= \infty\\ \lim_{x\to\infty} x &= \infty\end{aligned}\right\}$$ so by l'Hôpital, $\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$

"$\infty \cdot 0$"

(7) Evaluate $\lim_{x\to\infty} x^2 e^{-x}$.

$\lim_{x\to\infty} e^x = \infty$

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

l'Hôpital: $\lim_{x\to\infty} x^2 = \infty$, $\lim_{x\to\infty} e^x = \infty$

l'Hôpital: $\lim_{x\to\infty} 2x = \infty$, $\lim_{x\to\infty} e^x = \infty$

"$0 \cdot \infty$"

(8) Evaluate $\lim_{x\to 0^+} x \log x$.

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

l'Hôpital, $\lim_{x\to 0^+} \log x = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$

What if we try $\lim_{x\to 0^+} \left(\frac{x}{1/\log x}\right)$ "=" $\lim_{x\to 0^+} \frac{1}{-\frac{1}{x\log^2 x}} = -\lim_{x\to 0^+} x\log^2 x$

l'Hôpital: $\lim_{x\to 0} x = 0$, $\lim_{x\to 0} \frac{1}{\log x} = 0$

(l'Hôpital's rule doesn't have to help)

(9) Evaluate $\lim_{x\to 0}(2x+1)^{1/\sin x}$.

$\lim_{x\to 0} 2x+1 = 1$, $\lim_{x\to 0^\pm} \frac{1}{\sin x} = \pm\infty$

$(2x+1)^{1/\sin x} = e^{\log\left[(2x+1)^{1/\sin x}\right]} = e^{\left[\frac{\log(2x+1)}{\sin x}\right]} \xrightarrow[x\to 0]{} e^2$

$\lim_{x\to 0} \frac{\log(2x+1)}{\sin x} = \lim_{x\to 0} \frac{2/(2x+1)}{\cos x} = \frac{2}{2\cdot 0+1}\cdot\frac{1}{1} = 2$

l'Hospital

$\lim_{x\to 0} \log(2x+1) = \log 1 = 0$

$\lim_{x\to 0} \sin x = 0$

$\lim_{x\to 0} f(x) = e^{\lim_{x\to 0} \log f(x)}$

(10) Evaluate $\lim_{x\to\infty} x^n e^{-x}$.

Alternative

Let $f(x) = (2x+1)^{1/\sin x}$. Then $\log f(x) = \frac{1}{\sin x}\log(2x+1)$

so $\lim_{x\to 0} \log f(x) = \ldots = 2$

so $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\log f(x)} = e^{\lim_{x\to 0}\log f(x)} = e^2$

continuity of e^u

(11) Suppose $a > 0$. Evaluate $\lim_{x\to\infty} x^{-a}\log x$.