16. Minima and Maxima (4/11/2021)

Goals.

(1) Global and local extrema
(2) Critical and singular points
(3) Finding minima and maxima using differentiation
(4) Midterm!

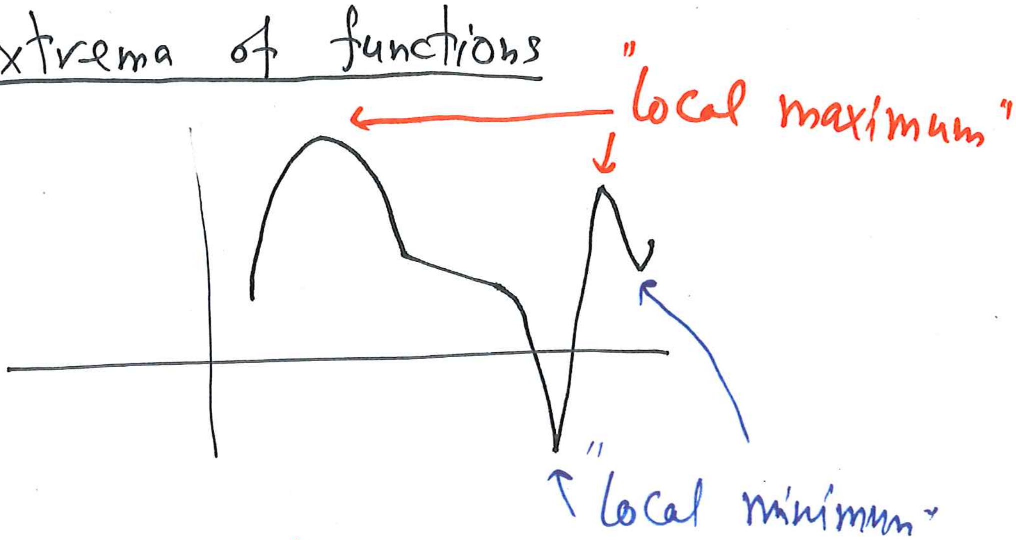
Last Time. Lagrange Remainder for Taylor expansion:

$$f(x) = \overbrace{f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}^{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^n}_{R_n(x)}$$

for some c between a, x.

Using estimates for $f^{(n+1)}(y)$ on $[a,x]$ can get estimates on $R_n(x)$, especially its magnitude.

Extrema of functions



local max/min = pt where nearby values are smaller/larger

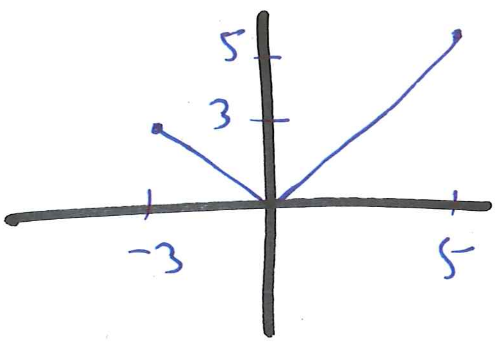
Global/absolute maximum/minimum: largest value in whole domain

Math 100 – WORKSHEET 16
MINIMA AND MAXIMA

1. Absolute minima and maxima by hand

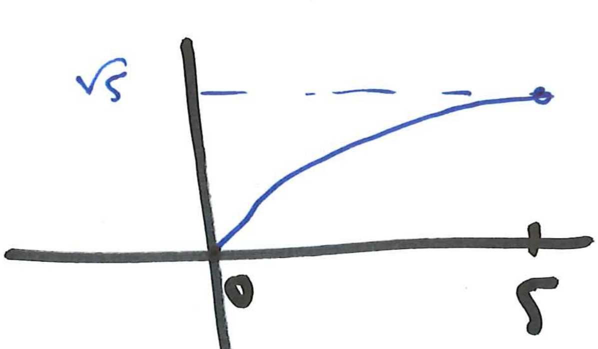
Theorem. *If f is continuous on $[a, b]$ it has an absolute maximum and minimum there.*

(1) Find the absolute maximum and minimum values of $f(x) = |x|$ on the interval $[-3, 5]$.



maximum value is 5, achieved at $x=5$
minimum value is 0, achieved at $x=0$

(2) Find the absolute maximum and minimum of $f(x) = \sqrt{x}$ on $[0, 5]$.



maximum value is $\sqrt{5}$, achieved at $x=5$
minimum value is 0, " " $x=0$

Date: 4/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Language: one maximum, many maxima
one minimum, many minima
one extremum, many extrema

Facts: ① if f cts on $[a,b]$, f achieves a global minimum & maximum

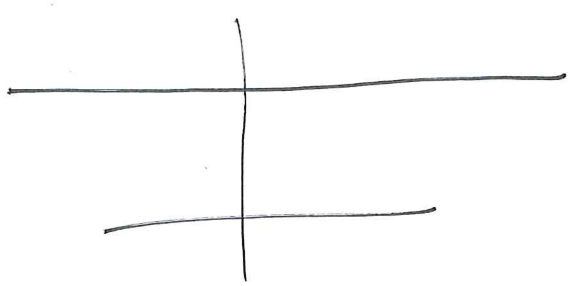
② If f has a local max/min at x_0 then $f'(x_0)=0$ if $f'(x_0)$ exists

$\Rightarrow$ If cts, extrema can only occur:

(1) at critical points: $f'(x_0)=0$

(2) at singular points: $f'(x_0)$ DNE

(3) at endpoints of the interval



$f(x)=c$

every point is a local max & a local min

(and $f'=0$)

2. Derivatives and local extrema

Theorem (Fermat). *If, in addition, f is defined near c (on both sides!), is differentiable at c, and has a local extremum at c then $f'(c) = 0$.*

Procedure

- Call c a *critical point* or *critical number* if $f'(c) = 0$, a *singular point/number* if $f'(c)$ does not exist.
- To find absolute maximum/minimum of a continuous function f defined on $[a, b]$:
  - Evaluate $f(c)$ at all critical and singular point.
  - Evaluate $f(a)$, $f(b)$.
  - Choose largest, smallest value.

(3) (Final, 2011) Let $f(x) = 6x^{1/5} + x^{6/5}$.

(a) Find the critical numbers and singularities of f.

Here $f'(x) = 6 \cdot \frac{1}{5} x^{-4/5} + \frac{6}{5} x^{1/5} = \frac{6}{5} \frac{1+x}{x^{4/5}}$

singular point at $\boxed{x=0}$

critical point at $\boxed{x=-1}$ (if $\frac{6}{5} \frac{1+x}{x^{4/5}} = 0$ then $1+x=0$)

(b) Find its absolute maximum and minimum on the internal $[-32, 32]$.

Extrema can only occur at singular, critical or endpoints

$f(0) = 6 \cdot 0^{1/5} + 0^{6/5} = 0$

$f(-1) = 6 \cdot (-1)^{1/5} + ((-1)^{1/5})^6 = -6 + 1 = -5$

$f(32) = 6 \cdot 2 + 2^6 = 76$

$f(-32) = -6 \cdot 2 + 2^6 = 52$

So absolute maximum is 76, achieved at $x = 32$

" minimum is -5, " " -1

(4) (Final, 2015) Find the critical points of $f(x) = e^{x^3 - 9x^2 + 15x - 1}$