21. Antiderivatives (25/11/2021)

Goals.

(1) Idea of inverse operation
(2) Antiderivatives by massaging
(3) Antiderivatives of sums

Last Time. l'Hôpital's rule

We have $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ $\boxed{\text{IF}}$ $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (or both infinite)

(also works if $x\to\infty$, $x\to-\infty$, or if $\lim_{x\to a}\frac{f'}{g'}$ exists in the extended sense only)

Question: What about $\lim_{x\to 0^+}\frac{\log x}{x}$?

$\lim_{x\to 0^+}\log x = -\infty$, $\lim_{x\to 0^+}\frac{1}{x} = \infty$, so $\lim_{x\to 0^+}\frac{1}{x}\cdot\log x = -\infty$

Watch out: $\lim_{x\to 0^+}\frac{1/x}{1} = \infty$: $\lim_{x\to 0^+}\log x = -\infty$
$\lim_{x\to 0^+} x = 0$ not same

Today: reverse differentiation

Math 100 – WORKSHEET 21
ANTIDERIVATIVES

1. Warmup: inverse operations

(1) (Multiplication)
  (a) Calculate: $7 \times 8 = 56$
  (b) Find (some) a, b such that $ab = 15$.

$15 = 3 \times 5 = 5 \times 3 = 1 \times 15 = 15 \times 1$

(1) multiplication is easy, but reverse mult. ("factoring") is hard (easy to check if 3×5 is a correct answer)

(2) mult. has one answer, several factorisations

(2) (Trig functions)
  (a) Calculate: $\sin \frac{\pi}{3} = \sqrt{3}/2$
  (b) Find all θ such that $\sin\theta = 1$.

$\ldots, \frac{\pi}{2} - 2\pi, \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots$

all θ are $\left\{ \frac{\pi}{2} + 2\pi k \mid k \in \mathbb{Z} \right\}$

(2') multiple solutions which are shifts of each other

Date: 25/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(3) Simple differentiation

(a) Find one f such that $f'(x) = 1$.

"a particular solution" $\rightarrow$ $f(x) = x$

also $x+1$, $x-\pi$, ..

(b) Find *all* such f.

"the general solution"

$\{x+c \mid c \in \mathbb{R}\}$ usually write $f(x) = x + C$

(c) Find the f such that $f(7) = 3$.

$f(7) = 7 + c$ so if $f(7) = 3$ we have $c = -4$, and

$\boxed{f(x) = x - 4}$ $\leftarrow$ "particular solution satisfying the condition $f(7)=3$".

Aside: why is $x+c$ the general solution?

Suppose $f'(x) = 1$. Let $g(x) = f(x) - x$. Then $g'(x) = 1 - 1 = 0$

[Fact: If $g'(x) \equiv 0$ then $g(x) \equiv C'$ for some]

Let $b \neq a$, then $\underbrace{\frac{g(b) - g(a)}{b - a}}_{\text{by MVT}} = g'(\xi) = 0$ & ξ between a, b

So $g(b) = g(a)$, i.e. $g(x) \equiv g(a) = C'$.

2. Antidifferentiation by massaging

(4) Find f such that $f'(x) = 2x^3$.

To start with, $(x^4)' = 4x^3$

so $\left(\frac{1}{2}x^4\right)' = \frac{1}{2}\cdot 4x^3 = 2x^3$ ✓.

(5) Find f such that $f'(x) = -\frac{1}{x}$.

Naive answer: $(\log x)' = \frac{1}{x}$ so $-\log x$ works

(really true that $\frac{d}{dx}(-\log x) = -\frac{1}{x}$)

defective: only works if $x > 0$

Correct: $(\log|x|)' = \frac{1}{x}$ so $f(x) = -\log|x|$ works

(6) Find all f such that $f'(x) = \cos 3x$.

$\frac{d}{d\theta}\sin\theta = \cos\theta$, so try $(\sin(3x))' = 3\cos 3x$ by chain rule

So $\left(\frac{1}{3}\sin(3x)\right)' = \cos 3x$ and the general solution is

$$f(x) = \frac{1}{3}\sin(3x) + C.$$

Want: g s.t. $g'(x) = \frac{1}{x}$ for $x<0$

let $y = -x$, $x = -y$. In this variable we have

$$\frac{dg}{dx} = -\frac{1}{y}, \quad \frac{dx}{dy} = -1 \qquad y>0.$$

So by chain rule $\frac{dg}{dy} = \frac{dg}{dx} \cdot \frac{dx}{dy} = -\frac{1}{y} \cdot (-1) = \frac{1}{y}$

So $g(x) = \log y = \log(-x) = \log|x|$

Fact: $\frac{d(\log|x|)}{dx} = \frac{1}{x}$ for all $x \neq 0$

$$\frac{d}{dy} \log(-y) = \frac{-1}{(-y)} = -\frac{1}{x} \qquad x<0$$

$$\frac{d}{dx} \log(x) = \frac{1}{x}$$

3. Combinations

(7) (Final, 2015) Find a function $f(x)$ such that $f'(x) = \sin x + \frac{2}{\sqrt{x}}$ and $f(\pi) = 0$.

particular solution

$(\cos x)' = -\sin x$, $(x^{1/2})' = \frac{1}{2}x^{-1/2}$. So $\left(-\cos x + 4x^{1/2}\right)' = \sin x + \frac{2}{\sqrt{x}}$

So $f(x) = -\cos x + 4x^{1/2} + c$ for some c general solution

Now $0 = f(\pi) = -\cos\pi + 4\pi^{1/2} + c$ So $c = -(1 + 4\sqrt{\pi})$

So $\boxed{f(x) = -\cos x + 4\sqrt{x} - (1 + 4\sqrt{\pi})}$

particular solution s.t. $f(\pi) = 0$

(8) (Final, 2016) Find the general antiderivative of $f(x) = e^{2x+3}$.

$\frac{d}{du}(e^u) = e^u$. So $(e^{2x+3})' = e^{2x+3} \cdot 2$

So $\boxed{\frac{1}{2}e^{2x+3} + C}$ is the general antiderivative

(9) Find f such that $f'(x) = \frac{6x^4 - 2x - 2}{x^2}$.

We note that $\frac{6x^4 - 2x - 2}{x^2} = 6x^2 - \frac{2}{x} - \frac{2}{x^2}$

And that $(x^3)' = 3x^2$, $(\log|x|)' = \frac{1}{x}$, $(\frac{1}{x})' = -\frac{1}{x^2}$

So $f(x) = 2x^3 - 2\log|x| + \frac{2}{x}$ works

($f(x) = 2x^3 - 2\log|x| + \frac{2}{x} + c$ also works)

(10) Find f such that $f'(x) = 2x^{1/3} - x^{-2/3}$ and $f(1000) = 5$.

$(x^{4/3})' = \frac{4}{3}x^{1/3}$, $(x^{1/3})' = \frac{1}{3}x^{-2/3}$, so

$f(x) = \frac{3}{2}x^{4/3} - 3x^{1/3} + c$ for some c

We have $5 = f(1000) = \frac{3}{2}10^4 - 3 \cdot 10 + c$

so $c = -15{,}000 + 35 = -14{,}965$

So $\boxed{f(x) = \frac{3}{2}x^{4/3} - 3x^{1/3} - 14{,}965}$

(11) Find f such that $f''(x) = \sin x + \cos x$, $f(0) = 0$ and $f'(0) = 1$.

since $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

$$f'(x) = -\cos x + \sin x + c$$

but $1 = f'(0) = -\cos 0 + \sin 0 + c$ so $\boxed{c = 2}$

so $f'(x) = -\cos x + \sin x + 2$

so $f(x) = -\sin x - \cos x + 2x + d$

but $0 = f(0) = -\sin 0 - \cos 0 + 2 \cdot 0 + d = -1 + d$

so $\boxed{d = 1}$

and $\boxed{f(x) = -\sin x - \cos x + 2x + 1}$