12. Exponetial growth and decay (21/10/2021)

Goals.

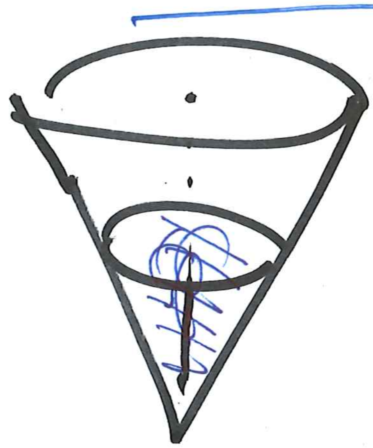
(1) More related rates
(2) Exponential growth
(3) Exponential decay: half-life
(4) Newton's law of cooling

Last Time.

Related rates: diff $F(x,y)=0$ wrt t.

Inverse trig: $\arcsin x$, $\arctan x$: use periodicity and reflection to compute $\arcsin(\sin\theta)$,

diff $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arctan x)' = \frac{1}{1+x^2}$.



conical tank of water

want to relate height and volume of the water.

Math 100 – WORKSHEET 12
EXPONENTIAL GROWTH AND DECAY

1. More Related Rates

(1) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

(0) read

(0) draw

(1) **names**

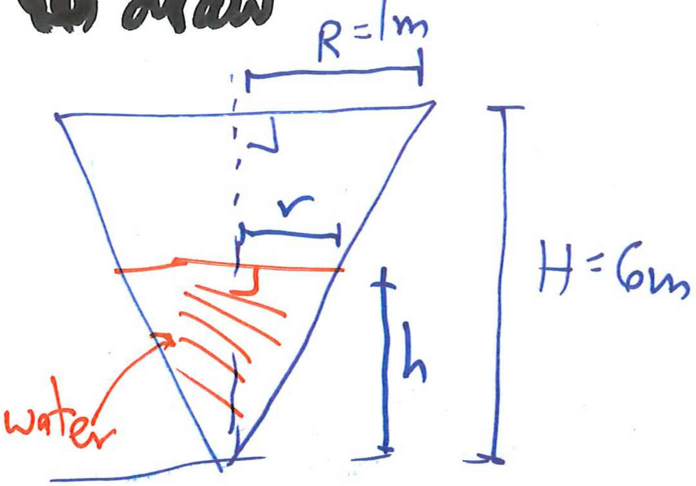
H = Height of tank

R = radius of (base) of tank

h = height of water

r = radius of (base) of water

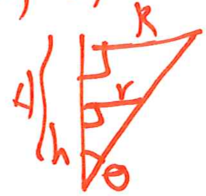
V = volume of water.



See CLP appendix for highschool facts

(2) **relations:** $V = \frac{1}{3}\cdot(\pi r^2)\cdot h$

$\frac{r}{h} = \frac{R}{H}$ ← similarity of triangles or tan θ



i.e. $r = \frac{1}{6}h = \frac{1}{6}h$

$\Rightarrow \quad V = \frac{1}{3}\pi\left(\frac{h}{6}\right)^2 h = \frac{\pi}{108}h^3$

(3) calculus: $\frac{dV}{dt} = \frac{\pi}{108}\cdot 3h^2 \cdot \frac{dh}{dt}$

(4) solution $\frac{dV}{dt} = 5\frac{m^3}{min}$

$\frac{dh}{dt} = \frac{5\cdot 36}{\pi\cdot 25}\frac{m}{min}\Big|_{h=5m}$

(b) The drain is unclogged and water begins to clear at the rate of $\frac{\pi}{4}\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of $1\text{m}/\text{min}$?

Still $\frac{dV}{dt} = \frac{\pi}{36} h^2 \frac{dh}{dt}$, this time solve for h given $\frac{dV}{dt}$, $\frac{dh}{dt}$.

Related Rates Summary

(0) Read problem: understand the idea, draw a picture if possible.

(1) Assign names:
- Choose axes, quantities of interest.
- Give a *name* to each quantity of interest.

(2) Function: write down the *relation* between the quantities of interest.

(3) Calculus: differentiate the relation using the chain rule

(4) Interpretation: solve the problem using the calculus result.
- Make *sanity checks* (area can't be negative, for example).

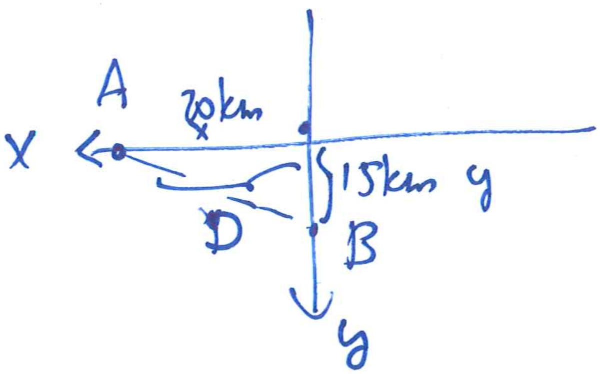
moving east at $5\frac{km}{h}$

(2) Two ships are travelling near an island. The first is located 20km due west of it, The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing if:

(a) The first ship is moving due ~~north~~ west at 5km/h.

here, $x^2+y^2=D^2$

(or $D=\sqrt{x^2+y^2}$)



so $2x\frac{dx}{dt}+2y\frac{dy}{dt}=2D\frac{dD}{dt}$

$x=20$km, $\frac{dx}{dt}=+5\frac{km}{h}$

$y=15$km, $\frac{dy}{dt}=7\frac{km}{h}$

(b) The same setting, but now the first ship is moving toward the island.

$\frac{dx}{dt}=-5\frac{km}{h}$

Exponential growth and decay

Idea: differential equation

often rate of change of a quantity is determined by its value.

Example: Say at time t, $y(t)$ people have a disease, in a short time every person can infect about $r\Delta t$ people. At time $t+\Delta t$ expect:

$$\approx \underbrace{y(t)}_{\text{already}} + \underbrace{y(t)\, r\Delta t}_{\text{new}} \quad \text{infected people}$$

so $y(t+\Delta t) \approx y(t) + y(t)\, r\Delta t$

so $\dfrac{y(t+\Delta t) - y(t)}{\Delta t} \approx r\, y(t)$

take $\Delta t \to 0$ get $\quad y'(t) = r\, y(t)$

This is an equation whose unknown, $y(t)$ is a function

Same equation describes population growth

Solution: $(e^t)' = e^t$, $(2^t)' = (\log 2)\cdot 2^t$

so $y(t) = Ce^{rt}$ for some constant C

Summary

simple growth / decay models have form $y' = ry$.

then $y(t)$ is exponential: $y(t) = C \cdot e^{rt}$.

notes $C = y(0)$, so it's the initial value

2. Exponential growth and decay

(3) Suppose[1] that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.

(a) At what time will there be 1000 opossums in BC? 10,000 opossums?

Let $N(t)$ = # opossums ~~@~~ t years after 1935.

Then $N(0) = 2$, $N(t) = 2 \cdot 3^t = 2 \cdot e^{(\log 3)t}$

if we want, switch to natural base

so $N(t) = 1000$ happens when

$2 \cdot e^{(\log 3)t} = 1000$ so $t = \frac{\log 500}{\log 3} = \log_3 500$

(b) Write a differential equation expressing the growth of the opossum population with time.

[1] See http://linnet.geog.ubc.ca/efauna/Atlas/Atlas.aspx?sciname=Didelphis%20virginiana

In exponential decay often use half-life:

time τ s.t. exactly $\frac{1}{2}$ of the quantity remains

ie the quantity is $y(t) = C \cdot 2^{-t/\tau}$

$\frac{t}{\tau}$ = # of half-lives elapsed

(4) A radioactive sample decays according to the law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km\,.$$

(a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?

(b) A 100-gram sample is left unattended for three days. How much of it remains?

(a) two half-lives are 10 hours so the half-life is 5 hrs

(b) $$\underbrace{2^{-72/5}}_{\text{fraction that remains}} \cdot \underbrace{100\,gr}_{\text{original quantity}} = e^{-\frac{\log 2}{5}\cdot 72} \cdot 100\,gr$$

$\approx 0.96\ gr.$

(5) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.

Let $N(t)$ = # of bacteria, ~~when~~ t measured in hours

Then $N(t) = N_0 \cdot 2^{t/4}$, N_0 = initial number

Have: $2000 = N_0 \cdot 2^{6/4}$ so $N_0 = \frac{2000}{2\sqrt{2}} = \frac{1000}{\sqrt{2}} = 500\sqrt{2}.$

3. Newton's Law of Cooling

Fact. *When a body of temperature T_0 is placed in an environment of temperature T_{env} the temperature difference $T(t) - T_{env}$ between the body and the environment decays exponentially. In other words, there is a (negative) constant k such that*

$$T' = k(T - T_{env}) \qquad\qquad T(t) - T_{env} = (T_0 - T_{env})e^{kt}\,.$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\mathrm{env}}$. Then
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = ky$$

Corollary. $\lim_{t\to\infty} y(t) = 0$ *so* $\lim_{t\to\infty} T(y) = T_{env}$.

(6) (Final, 2010) When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.

let $y(t) = T_{apple}(t) - T_{room}$; $y(0) = -16°C$

$y(30\,min) = -8°C$ in 30min, $y(t)$ decreased by $\frac{1}{2}$

so $y(90\,min) = (\frac{1}{2})^3 \cdot (-16°C) = -2°C$, $T_{apple}(90\,min) = 17°C$

By NLC

$y(t) = -16°C \cdot e^{kt}$, $y(30) = -8°C = -16° \cdot e^{30k}$ so $e^{30k} = \frac{1}{2}$

so $k = \frac{1}{30} \log \frac{1}{2} = -\frac{1}{30} \log 2$. Then $y(90) = -16°C \cdot e^{-\frac{1}{30} \log 2 \cdot 90}$ ..

(b) Determine the time when the temperature of the apple is $16°C$.

(c) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.

(7) (Final, 2013) A bottle of soda pop at room temperature ($70°F$) is placed in the refrigerator where the temperature is $40°F$. After half han hour the bottle has cooled to $60°F$. When will it reach $50°F$?

(6) (Final, 2010) When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.

Let $y(t) = T_{apple}(t) - T_{room}$ $\quad y(0) = 3° - 19° = -16°C$

$y(30) = 11° - 19° = -8°C$ $\quad$ so $y(30) = \frac{1}{2} y(0)$

so $y(90) = \left(\frac{1}{2}\right)^3 y(0) = -2°C$, so $T(90) = 17°C$

or: $y(t) = y(0) \cdot e^{kt} = -16°C \cdot e^{kt}$ $\quad$ so $-8 = y(30) = -16 e^{k \cdot 30}$

so $k = \frac{1}{30} \log\left(\frac{1}{2}\right) = -\frac{\log 2}{30}$, so $y(90) = -16°C \cdot e^{-\frac{90}{30} \log 2}$

(b) Determine the time when the temperature of the apple is $16°C$.

not $16 e^{kt} = 16$ but $-16 e^{kt} = -3$

(c) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.