5. The intermediate value theorem (23/9/2021)

Goals:

(1) The IVT
  (a) With given endpoints
  (b) Free-form (you find endpoints)
(2) (if there's time) The derivative

Last Time. **Continuity**: f is cts at a if

$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$. ("no break in graph")

Promise: If f is defined by formula at & near a then f is cts at a.

Two ideas: (1) check continuity by computing limits ("gluing of functions")
(2) Use continuity to evaluate limits

Today: **Theorem**: If f is cts on $[a,b]$ then f takes every value between $f(a)$, $f(b)$.

("no jumps")

Plan: (1) conceptual difficulty

(2) easy application (basic idea) cts $f(a), f(b)$

(3) more ideas: setting up problem, playing an endgame

(4) what if a, b not given? finding a,b inequalities

Difficulty: Method is about "existence".

Worksheet (1)

Example: Person A starts at bottom of the hill,
" B " " top " " ".

They walk along the same path, switching places. Show that they meet.

Let $f(t)$ = height of person A at time t
$g(t)$ = " " " B " " t } giving names

Want time c s.t. $f(c) = g(c)$ c = name of meeting time

$\iff f(c) - g(c) = 0.$

So let $h(t) = f(t) - g(t)$ (want c s.t. $h(c) = 0$)

1. The Intermediate Value Theorem

(1) Show that $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \leq x \leq 1$.

f is defined by formula, hence cts.

$f(0) = 1$, $f(1) = -2$, and $-2 < 0 < 1$.

By the IVT there is c, $0 < c < 1$ s.t. such that $f(c) = 0$.

Date: 23/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$h(\text{morning}) = -H$, H: height of the hill
$h(\text{evening}) = H$ so somewhere in between
there was a time where $h(c)=0$

Worksheet (2)

(2) (Final 2011) Let $y = f(x)$ be continuous with domain $[0,1]$ and range in $[3,5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

Want $0 \le c \le 1$ s.t. $f(c) = 2c + 3$

Let $g(x) = f(x) - (2x+3)$ ← subtraction (want c s.t. $g(c) = 0$)

$2x+3$ is cts (polynomial), f is cts by hypothesis,

so $g(x) = f(x) - (2x+3)$ is cts ← check continuity

$g(0) = f(0) - 3 \in [0,2]$, so $g(0) \ge 0$ $(f(0) \in [3,5])$

$g(1) = f(1) - 5 \in [-2,0]$ so $g(1) \le 0$ $(f(1) \in [3,5])$ ← evaluate at endpoints

By the IVT there is c, $0 \le c \le 1$ s.t. $g(c) = 0$

← invoke IVT

So $f(c) - (2c+3) = 0$ so $f(c) = 2c+3$, i.e. the graphs intersect at c } endgame

(4) (Final 2015) Show that the equation $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions.

(3) $\sin x = x + 1$ has a solution.

Let $f(x) = \sin x - (x+1)$

f is defined by formula, hence cts

$f(0) = -1$, $f(-\frac{\pi}{2}) = -1 - (-\frac{\pi}{2} + 1) = +\frac{\pi}{2} - 2$

$\leq \frac{4}{2} - 2 = 0$

$f(\frac{\pi}{2}) = 1 - (\frac{\pi}{2} + 1) = -\frac{\pi}{2}$

$f(-1) = -\sin 1$

$f(-\pi) = 0 - (-\pi + 1) = \pi - 1 > 0$

By IVT there is c, $-\pi < c < 0$ s.t. $f(c) = 0$

i.e. $\sin c = c + 1$.

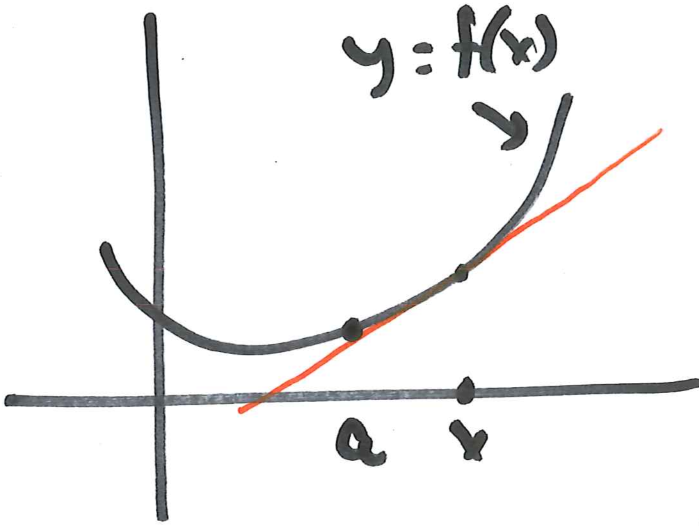
Or: $f(100) = \sin 100 - 101 \leq -100 < 0$

$f(-100) = \sin(-100) - (-99) = 99 - \sin 100 \geq 98 > 0$

By IVT there is c, $-100 < c < 100$ s.t. ..

The Derivative

Recall lecture 1



To find line tangent to graph of f at $(a, f(a))$, choose point x, find line through $(a, f(a))$, $(x, f(x))$: has slope: $\frac{f(x)-f(a)}{x-a}$

then take limit as $x \to a$

Def: We call $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ the derivative of f at a, write $f'(a)$. (If the limit exists, then we say f is differentiable at a)

Why care?? (1) this is why $\frac{d}{dx} x^2 = 2x$