Lior Silberman's Set Theory: Problem Set 5

Not everything requires choice

1. Let A be a set and suppose. $f\colon \omega \to A$ is surjective. Show that there is an injective function $f\colon A \to \omega$.
2. Prove (without the AC!) that for every finite set A if every $a \in A$ is non-empty there is $f\colon A \to \bigcup A$ with $f(a) \in a$ for all $a \in A$.

Cardinality and cardinal arithmetic

3. For $f\colon I \to \mathbb{R}_{\geq 0}$ define $\sum_{i\in I} f(i) = \sup\left\{\sum_{j\in J} f(j) \mid J \subset I \text{ finite}\right\}$. Suppose $f(I) < \infty$. Show that $\{i \mid f(i) \neq 0\}$ is countable.
4. Let A be an infinite set.
   (a) A *finite sequence* in A is a function $f\colon n \to A$ with $n \in \omega$. Show that there is a "set of all finite sequences in A".
   (b) Show that the set of all finite sequences in A has the same cardinality as A.
   (c) Review the proof that the set of algebraic numbers is countable.

5. For a set A let $S_A = \{f\colon A \to A \mid f \text{ is bijective}\}$ by the *symmetric group* of A. Show that if A is infinite we have $|S_A| = 2^{|A|}$ (in terms of cardinal numbers, this reads $\kappa! = 2^\kappa$).
6. Let $(A, <_A)$ and $(B, <_B)$ be two well-ordered sets. Show that exactly one of the following is true:
   (1) There is a isomorphism of ordered sets $f\colon A \to B$.
   (2) There is $a \in A$ so that $(B, <_B)$ is order-isomorphic to $(\operatorname{seg} a, <_A)$.
   (3) There is $b \in B$ so that $(A, <_A)$ is order-isomorphic to $(\operatorname{seg} b, <_B)$.