Math 100 – SOLUTIONS TO WORKSHEET 15
TAYLOR REMAINDER ESTIMATES

1. Review: Taylor expansion

(1) Estimate $(4.1)^{3/2}$ using a linear and a quadratic approximation.

Solution: Let $f(x) = x^{3/2}$ so that $f'(x) = \frac{3}{2}x^{1/2}$ and $f''(x) = \frac{3}{4}x^{-1/2}$. Then $f(4) = 8$, $f'(4) = \frac{3}{2} \cdot 2 = 3$ and $f''(4) = \frac{3}{8}$. The linear approximation is $T_1(x) = 4 + 3(x-4)$, the quadratic approximation is $T_2(x) = 4 + 3(x-4) + \frac{3}{16}(x-4)^2$ and in particular

$$\begin{aligned} T_1(4.1) &= 4 + 3 \cdot 0.1 = 4.3\,. \\ T_2(4.1) &= 4 + 3 \cdot 0.1 + \frac{3}{16} \cdot (0.1)^2 = 4.3 + \frac{3 \cdot 625}{10^6} \\ &= 4.301875\,. \end{aligned}$$

(2) The third-order expansion of $h(x)$ about $x = 2$ is $3 + \frac{1}{2}(x-2) + 2(x-2)^3$. What are $h'(2)$ and $h''(2)$?

Solution: $h'(2) = \frac{1}{2}$ and $\frac{h''(2)}{2!} = 0$ (no quadratic term) so $h''(2) = 0$.

(3) (Final, 2016) Find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

Solution: Let $f(x) = \sin x$. Then $f'(x) = \cos x$, $f^{(2)}(x) = -\sin x$ and $f^{(3)}(x) = -\cos x$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$ and the third-order expansion of $\sin x$ is $0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 = x - \frac{1}{6}x^3$. We then have, correct to third order, that

$$(x+1)\sin x \approx (x+1)\left(x - \frac{1}{6}x^3\right) = x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 \approx x + x^2 - \frac{1}{6}x^3\,.$$

Solution: Let $g(x) = (x+1)\sin x$. Then $f'(x) = \sin x + (x+1)\cos x$, $f''(x) = 2\cos x - (x+1)\sin x$, $f^{(3)}(x) = -3\sin x - (x+1)\cos x$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, $f^{(3)}(0) = -1$ and

$$T_3(x) = 0 + \frac{1}{1!}x + \frac{2}{2!}x^2 - \frac{1}{6!}x^3 = x + x^2 - \frac{1}{6}x^3\,.$$

2. Error estimate 1

Let $R_1(x) = f(x) - T_1(x)$ be the *remainder*. Then there is c between a and x such that

$$R_1(x) = \frac{f^{(2)}(c)}{2!}(x-a)^2$$

(4) Estimate the error in the linear approximations to $(4.1)^{3/2}$.

Solution: By the Lagrange remainder formula

$$R_1(4.1) = f(4.1) - T_1(4) = \frac{1}{2!} \cdot \frac{3}{4}c^{-1/2}\,(0.1)^2$$

for some $4 \le c \le 4.1$. The error is therefore positive ($T_1(4.1)$ is an *overestimate*) and its magnitude is at most $\frac{3}{800} \cdot \frac{1}{4^{1/2}} = \frac{3}{1600}$.

(5) (Final, 2012) Show $-\frac{5}{32} \le \log\left(\frac{8}{9}\right) \le -\frac{1}{9}$ using the linear approximation to $f(x) = \log\left(1 - x^2\right)$.

Solution: We have $f'(x) = -\frac{2x}{1-x^2}$ and $f''(x) = -\frac{2(1-x^2) - 2x(-2x)}{(1-x^2)^2} = -\frac{2+2x^2}{(1-x^2)^2} = -2\frac{1+x^2}{(1-x^2)^2}$. Since $f(0) = 0$ and $f'(0) = 0$ the linear approximation to $\log\left(\frac{8}{9}\right) = f\left(\frac{1}{3}\right)$ is $T_1\left(\frac{1}{3}\right) = 0 + 0 \cdot \frac{1}{3} = 0$. The error satisfies

$$R_1\left(\frac{1}{3}\right) = \frac{1}{2!} \cdot (-2) \cdot \frac{1+c^2}{(1-c^2)^2}\left(\frac{1}{3}\right)^2$$

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for some $0 \le c \le \frac{1}{3}$. Now the expression $\frac{1+c^2}{(1-c^2)^2}$ is increasing for $0 \le c \le \frac{1}{3}$ (clear for the numerator, and for the denominator note that $1-c^2$ is decreasing). It follows that $1 \le \frac{1+c^2}{(1-c^2)^2} \le \frac{1+\frac{1}{9}}{(8/9)^2} = \frac{90}{8^2} = \frac{45}{32}$. We therefore have

$$-1 \cdot \frac{1}{9} = -\frac{1}{9} \ge R_1\left(\frac{1}{3}\right) \ge -\frac{45}{32} \cdot \frac{1}{9} = -\frac{5}{32}$$

3. Higher order error estimates

Let $R_n(x) = f(x) - T_n(x)$ be the *remainder*. Then there is c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

(6) Estimate the magnitude of the error in the quadratic approximation to $(4.1)^{3/2}$.

Solution: We have $f^{(3)}(x) = -\frac{3}{16}x^{-3/2}$. Thus

$$R_2(x) = -\frac{1}{3!} \cdot \frac{3}{16}c^{-3/2}(0.1)^3 = -\frac{1}{3200}c^{-3/2}$$

for some $4 < c < 4.1$. Now the magnitude of this function decreases with c, so

$$|R_2(x)| \le \frac{1}{32000} \cdot 4^{-3/2} = \frac{8}{32000} = \frac{1}{4000} = 0.00025\,.$$

(7) (Quiz, 2015) Consider a function f such that $f^{(4)}(x) = \frac{\cos(x^2)}{3-x}$. Show that, when approximating $f(0.5)$ using its third-degree MacLaurin polynomial, the absolute value of the error is less than $\frac{1}{500}$.

Solution: The Lagrange remainder formula shows that

$$R_3(0.5) = \frac{1}{4!} \cdot \frac{\cos(c^2)}{3-c} \cdot (0.5)^4$$

for some $0 < c < 0.5$. Then $\left|\cos(c^2)\right| \le 1$ and $\left|\frac{1}{3-c}\right| \le \frac{1}{3-0.5} = \frac{2}{5}$. We therefore have

$$\left|R_3\left(\frac{1}{2}\right)\right| = \frac{1}{24} \cdot \frac{2}{5} \cdot \frac{1}{16} = \frac{1}{120 \cdot 8} = \frac{1}{960} < \frac{1}{500}\,.$$

(8) (Final, 2012) Show that for all $-1 \le x \le 1$ we have

$$0 \le \cos(x) - \left(1 - \frac{x^2}{2}\right) \le \frac{1}{24}\,.$$

Solution: Let $f(x) = \cos x$. Then $f'(x) = -\sin x$, $f^{(2)}(x) = -\cos x$, $f^{(3)}(x) = \sin x$, $f^{(4)}(x) = \cos x$. Thus $f(0) = 1$, $f'(0) = 0$, $f^{(2)}(0) = -1$ and $f^{(3)}(0) = 0$. The third-order MacLaurin polynomia of f is therefore

$$T_3(x) = 1 + 0x - \frac{1}{2!}x^2 + 0x^3 = 1 - \frac{1}{2}x^2\,.$$

We therefore have $\cos(x) - \left(1 - \frac{x^2}{2}\right) = f(x) - T_3(x) = R_3(x)$. By the Lagrange form there is c between 0 and x (in particular, $-1 < c < 1$) so that

$$R_3(x) = \frac{1}{4!}f^{(4)}(c) \cdot x^4 = \frac{\cos c}{24} \cdot x^4\,.$$

Now x^4 is always positive and $\cos c$ is positive on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since $\pi > 3$,$\frac{\pi}{2} > 1.5 > 1$ and $\cos c$ is positive, so $R_3(x) > 0$ for all $x \in [-1,1]$. On the other hand $\cos c \le 1$ for all c and $x^4 \le 1$ if $|x| \le 1$. We therefore have $R_3(x) \le \frac{1}{24} \cdot 1 = \frac{1}{24}$.