# Math 100 – SOLUTIONS TO WORKSHEET 12
# EXPONENTIAL GROWTH AND DECAY

## 1. Exponentials

(1) Suppose[1] that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.

  (a) At what time will there be 1000 opossums in BC? 10,000 opossums?
  **Solution:** After $t$ years there are $2 \cdot 3^t$ opossums, so there will be 1000 opossums after $\frac{\log 500}{\log 3} \approx 5.7$ years and $10,000$ opossums after $\frac{\log 5000}{\log 3} \approx 7.8$ years.

  (b) Write a differential equation expressing the growth of the opossum population with time.
  **Solution:** Since $\frac{\mathrm{d}}{\mathrm{d}t}3^t = \log 3 \cdot 3^t$ the differential equation is $\frac{\mathrm{d}N}{\mathrm{d}t} = \log 3 \cdot N$.

(2) A radioactive sample decays according to the law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km\,.$$

  (a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?

  (b) A 100-gram sample is left unattended for three days. How much of it remains?
  **Solution:** If two halvings happened in 10 hours, the half-life is 5 hours. Accordingly after three days we will have $\frac{72}{5}$ half-lives, so the remaining mass will be

$$100 \cdot 2^{-\frac{72}{5}} = 100 \cdot \exp\left(-\frac{72}{5}\log 2\right) \approx 0.005\text{g}\,.$$

(3) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.
  **Solution:** After 6 hours we've had 1.5 doublings, so the original size is $\frac{2000}{2^{1.5}} = 1000\frac{\sqrt{2}}{2} \approx 707$.

## 2. Newton's Law of Cooling

**Fact.** *When a body of temperature $T_0$ is placed in an environment of temperature $T_{env}$, the rate of change of the temperature $T(t)$ is* negatively proportional *to the temperature difference $T - T_{env}$. In other words, there is a (negative) constant $k$ such that*

$$T' = k(T - T_{env})\,.$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\text{env}}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = ky$$

so there is $C$ for which

$$y(t) = Ce^{kt}\,.$$

Solving for $T$ we get:

$$T(t) = T_{\text{env}} + Ce^{kt}\,.$$

Setting $t = 0$ we find $T_{\text{env}} + C = T_0$ so $C = T_0 - T_{\text{env}}$ and

$$T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{kt}\,.$$

**Corollary.** $\lim_{t\to\infty} y(t) = 0$ *so* $\lim_{t\to\infty} T(y) = T_{env}$.

(1) (Final, 2010) When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

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[1] See http://linnet.geog.ubc.ca/efauna/Atlas/Atlas.aspx?sciname=Didelphis%20virginiana

(a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.
**Solution:** Let $T(t)$ be the temperature of the apple $t$ minutes after it was taken from the refrigerator, and let $y(t) = T(t) - 19$ be the temperature difference. Newton's law of cooling provides that $y(t)$ decays exponentially at constant rate. We are given that $y(0) = -16°C$ and that $y(30) = -8°C$ so the temperature difference was halved after 30 minutes. By 90 minutes there would be two further halvings, so $y(90) = -2°C$ and $T(90) = -2 + 19 = 17°C$.

(b) Determine the time when the temperature of the apple is $16°C$.
**Solution:** We are asked when the temperature difference will be $-3°C$. Since the temperature difference satisfies the law $y(t) = -16°C \cdot 2^{-t/30}$ we need to find $t$ so that

$$-3 = -16 \cdot 2^{-t/30}$$

that is

$$2^{t/30} = \frac{16}{3} .$$

Taking logarithms of both sides we have

$$\frac{t}{30} \log 2 = \log 16 - \log 3$$

so that the apple reaches $16°C$ at time

$$t = 30 \cdot \frac{\log 16 - \log 2}{\log 2} \text{ minutes.}$$

(c) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.
**Solution:** We are asked when the temperature difference will be $-3°C$. Since the temperature difference satisfies the law $y(t) = -16°C \cdot 2^{-t/30}$ we need to find $t$ so that

$$-3 = -16 \cdot 2^{-t/30}$$

that is

$$2^{t/30} = \frac{16}{3} .$$

Taking logarithms of both sides we have

$$\frac{t}{30} \log 2 = \log 16 - \log 3$$

so that the apple reaches $16°C$ at time

$$t = 30 \cdot \frac{\log 16 - \log 2}{\log 2} \text{ minutes.}$$

The temperature of the apple at time $T$ is $T(t) = 19°C - 16°C \cdot 2^{-t/30} = T = 19°C - 16°C \cdot e^{-\frac{\log 2}{30} t}$ .We therefore have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\log 2}{30} (T - 19) .$$

(2) (Final, 2013) A bottle of soda pop at room temperature ($70°F$) is placed in the refrigerator where the temperature is $40°F$. After half han hour the bottle has cooled to $60°F$. When will it reach $50°F$?

**Solution:** Let $T(t)$ be the temperature of the soda $t$ minutes after it was put in the fridge, and let $y(t) = T(t) - 40°F$ be the temperature difference. Then we are given that $y(0) = 30°F$ and that $y(30) = 20°F$. Newton's law of cooling provides that $y(t)$ decays exponentially at constant rate, so we conclude that $y(t) = 30°F \cdot \left(\frac{2}{3}\right)^{t/30}$ . We are asked for the time when $T(t) = 50°F$, that is when $y(t) = 10°F$. That time therefore satisfies:

$$10 = 30 \cdot \left(\frac{2}{3}\right)^{t/30} ,$$

that is

$$3 = \left(\frac{3}{2}\right)^{t/30}.$$

Taking logarithms we find

$$\log 3 = \frac{t}{30} \log \frac{3}{2}$$

so

$$t = 30 \frac{\log 3}{\log 3 - \log 2} \text{ minutes.}$$