Math 100, lecture 20 : Review (12/11/2019)

Plan: (1) Limits
(2) Related rates
(3) Lagrange remainder
(4) questions? – IVT
– MVT

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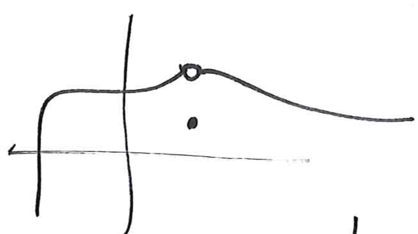
① Limits

"Def": $\lim_{x\to a} f(x)$ is the value f "would like to have" at $x=a$. Can be different from $f(a)$.

Example:



If the function isn't commited to a single value, $\lim_{x\to a} f(x)$ does not exist ("DNE")

* usually arise from a "limiting process".

Evaluating limits: (1) If f def by formula which makes sense at a, then $\lim_{x\to a} f(x) = f(a)$

(2) Arithmetic of limits: compute $\lim_{x\to a}(f+g) = \lim_{x\to a} f + \lim_{x\to a} g$ etc.

(3) Algebraic simplification: maybe $f(x) = g(x)$ near a, but g is defined at a: $\lim_{x\to 2}\frac{(x-3)(x-2)}{(x-2)} = \lim_{x\to 2} x-3 = -1$

(5) sometimes, limits are about cancellation,

trick: $\sqrt{a} - \sqrt{b} = (\sqrt{a} - \sqrt{b}) \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$

allows us to study cancellation of square roots

Example: $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2+2x}\right) = \lim_{x\to\infty} \frac{(x^2+3x)-(x^2+2x)}{\sqrt{x^2+3x}+\sqrt{x^2+2x}} =$

(4) asymptotics $= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+3x}+\sqrt{x^2+2x}} =$

$$= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{3}{x}} + x\sqrt{1+\frac{2}{x}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{3}{x}}+\sqrt{1+\frac{2}{x}}} = \frac{1}{2}$$

extract growth of each term

(6) squeeze thm: if $f(x) \le g(x) \le h(x)$ and f, h tend to the same limit at $x=a$, so does g.

Example: Def'n of derivative is $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(1) Let $f(x) = \frac{x^2+1}{x+2}$, then $\frac{f(x+h)-f(x)}{h} = \frac{\left(\frac{(x+h)^2+1}{(x+h)+2} - \frac{x^2+1}{x+2}\right)}{h} =$

$$= \frac{1}{h}\left(\frac{(x^2+2xh+h^2+1)(x+2)-(x^2+1)(x+h+2)}{(x+h+2)(x+2)}\right)$$

$$= \frac{\cancel{(x^2+1)(x+2)} + (2xh+h^2)(x+2) - \cancel{(x^2+1)(x+2)} - h(x^2+1)}{h(x+h+2)(x+2)}$$

$$= \frac{\cancel{h}(2x+h)(x+2) - \cancel{h}(x^2+1)}{\cancel{h}(x+h+2)(x+2)} = \frac{(2x+h)(x+2)-(x^2+1)}{(x+h+2)(x+2)}.$$

So: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(2x+h)(x+2)-(x^2+1)}{(x+h+2)(x+2)} = \frac{2x(x+2)-x^2-1}{(x+2)(x+2)} =$

$$= \frac{x^2+4x-1}{(x+2)^2}.$$

Example from midterm

Let $f(x) = \begin{cases} 2-x & x<2 \\ -g(x-2) & x \geq 2 \end{cases}$. Then $f(2) = -g(2-2) = -g(0) \stackrel{g(0)=0}{=} 0$.

We evaluate the limit $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ by considering $h>0$, and $h<0$ separately.

$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{2-(2+h)-0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$

f(2) not 2-2

$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{-g(2+h-2)-0}{h} = \lim_{h\to 0^+} \frac{-g(h)}{h} = -\lim_{h\to 0^+} \frac{g(h)}{h}.$

Now for $h>0$, $1-h \leq \frac{g(h)}{h} \leq 1$ and $\lim_{h\to 0^+}(1-h) = \lim_{h\to 0^+} 1 = 1$

so by the squeeze thm, $\lim_{h\to 0^+} \frac{g(h)}{h} = 1$ as well, and

$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = -\lim_{h\to 0^+} \frac{g(h)}{h} = -1.$

Since both limits agree we have $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = -1$ and f is differentiable at $x=2$.

Related rates

In general, have two quantities A, B, depend on a parameter, t. Have relationship $F(A,B)=0$
Diff this wrt t, get (using chain rule) relation connecting $A, B, \frac{dA}{dt}, \frac{dB}{dt}$. Given two of the four values (not just A, B) can compute other two.

Often in exams: relation is stated in words; must find $F(A,B)$ ourselves

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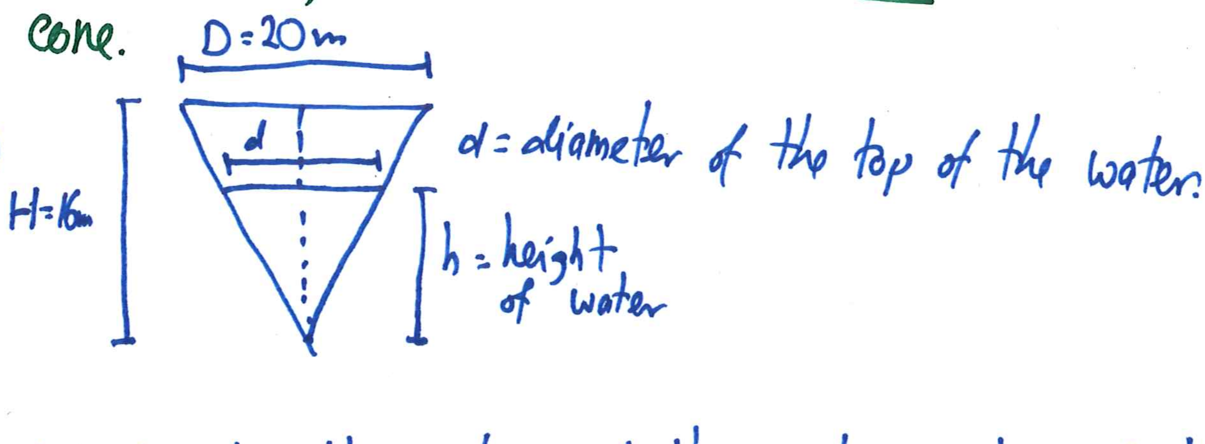
Example: 2015 final, problem 8

We are given information about rate of change of the volume of the water in the cone. Asked to find the rate of change of the height.

* the water will also take a conical shape.
* to relate height to volume need radius of the base of this cone.

(a)



(b) Let V be the volume of the water. Then $V=\frac{1}{3}\pi\left(\frac{d}{2}\right)^2 \cdot h$.
The ~~[illegible]~~ isoceles triangles formed (in the cross-section) by the water and the tank share ~~their~~ their angle, and are therefore similar.

Thus: $\frac{d}{h} = \frac{D}{H}$, so $d = \frac{D}{H}h$.

We then have $V = \frac{\pi}{12}\left(\frac{D}{H}\right)^2 h^2 \cdot h = \frac{\pi}{12}\left(\frac{D}{H}\right)^2 \cdot h^3$

Differentiating wrt t, we get

$$\frac{dV}{dt} = \frac{\pi}{12}\left(\frac{D}{H}\right)^2 \cdot 3h^2 \cdot \frac{dh}{dt}$$

$\frac{dV}{dt} = \frac{2m^3}{min}$ (water is flowing in)

$$\text{so } \frac{dh}{dt} = \frac{4}{\pi h^2}\left(\frac{H}{D}\right)^2 \frac{dV}{dt} = \frac{4}{\pi \cdot 10^2} \cdot \left(\frac{16}{20}\right)^2 \cdot 2 = \frac{32}{625\pi} \frac{m}{min}$$

$h = 10m$

The Lagrange remainder for Taylor expansion

Recall: The nth order Taylor expansion of f about a is the polynomial

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(Idea: extrapolate to other values of x information about f at a)

Remainder: $R_n(x) = f(x) - T_n(x)$

Thm: suppose $f^{(n+1)}(x)$ exists. Then $R_n(x)$ "looks like" the next term:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$ for some c between a, x.

Often care about "how big is $R_n(x)$ at most?"

Question: why can't we just take larger of $f^{(n+1)}(a), f^{(n+1)}(x)$?

Answer: because functions need not have their extrema at endpoints.

Example: 2015 final, question 11(b)

Given $f^{(3)}(x) = \dfrac{x\sin x + x^2\cos x}{10-x^2}$, we need to bound $R_2(1)$ for expansion about $x=0$.

By Lagrange form of the remainder,

$$R_2(x) = \frac{f^{(3)}(c)}{3!}(x-0)^3$$ for some $0<c<x$, here

$$R_2(1) = \frac{f^{(3)}(c)}{6}(1-0)^3 = \frac{1}{6} f^{(3)}(c)$$

Now $|f^{(3)}(c)| \le \left|\dfrac{c\cdot \sin c + c^2\cos c}{10-c^2}\right|$

here, $c \le 1$, $\sin c \le 1$, $c^2 \le 1$, $\cos c \le 1$, $10-c^2 \ge 9$

so $|f^{(3)}(c)| \le \dfrac{1\cdot 1 + 1\cdot 1}{9} \le \dfrac{2}{9}$, $|R_2(1)| \le \dfrac{1}{6}\cdot\dfrac{2}{9} \le \dfrac{1}{27} < \dfrac{1}{25}$