18. THE MVT AND THE GRAPH OF THE FUNCTION (5/11/2019)

Goals.

(1) More MVT examples
(2) Implications for the shape of the graph:
  (a) Increasing and decreasing functions
  (b) Concave and convex functions
(3) Curve sketching

Last Time.

MVT: If f' exists (on (a,b), f cts at a,b) then there is c so that $a<c<b$ and $\frac{f(b)-f(a)}{b-a} = f'(c)$

Use: info about f' $\rightsquigarrow$ info about f

Note: Also says: $f(x) = f(a) + f'(c)(x-a)$
(i.e. this is the case $n=0$ of the ~~Lg~~ Lagrange remainder)

Question: if we use $f'(a)$ instead, must it always bound $f(c)$?
(e.g. write $f(x) \le f(a) + f'(a)(x-a)$)

$\Leftrightarrow$ must $f'(a), f'(b)$ bracket $f'(c)$?

Worksheet 1

1. Applying the MVT

(1) Suppose $f'(x) = \frac{e^x}{x+\pi}$ for $0 \leq x \leq 2$. Give an upper bound for $f(2) - f(0)$.

By the MVT (f' exists on $[0,2]$), $\frac{f(2)-f(0)}{2-0} = f'(c) = \frac{e^c}{c+\pi}$ for some c between $(0,2)$,

so $f(2) - f(0) = \frac{2e^c}{c+\pi} \leq \frac{2e^2}{\pi}$

$e^c \leq e^2$ if $c \leq 2$

$\frac{1}{c+\pi} \leq \frac{1}{\pi}$ if $c \geq 0$

(2) (Final, 2015) Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at most two solutions.

Let $f(x) = 2x^2 - 3 + \sin x + \cos x$. Then f is ~~everywhere diff.~~ differentiable everywhere. If $f(a) = f(b) = 0$ $a<b$ then by MVT there is c between a,b s.t. $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$.

Now if $a<b<c$ are ~~two~~ three distinct solution to $f(x)=0$ Then have d between a,b s.t. $f'(d)=0$ and we have e, between b,c s.t. $f'(e)=0$. Now $d<b<e$ so between d,e have point x s.t. $(f')'(x) = 0$. But $(f')'(x) = f''(x) = 4 - \sin x - \cos x \geq 4-1-1 \geq 2 > 0$

That's a contradiction, so f can't have had 3 distinct zeroes.

(3) Suppose f satisfies the hypotheses of the MVT and that $f'(x) > 0$ for all $x \in (a, b)$. Show that $\frac{f(b)-f(a)}{b-a} > 0$, and hence that $f(b) > f(a)$.

By the MVT, there is c between a, b s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c) > 0$$

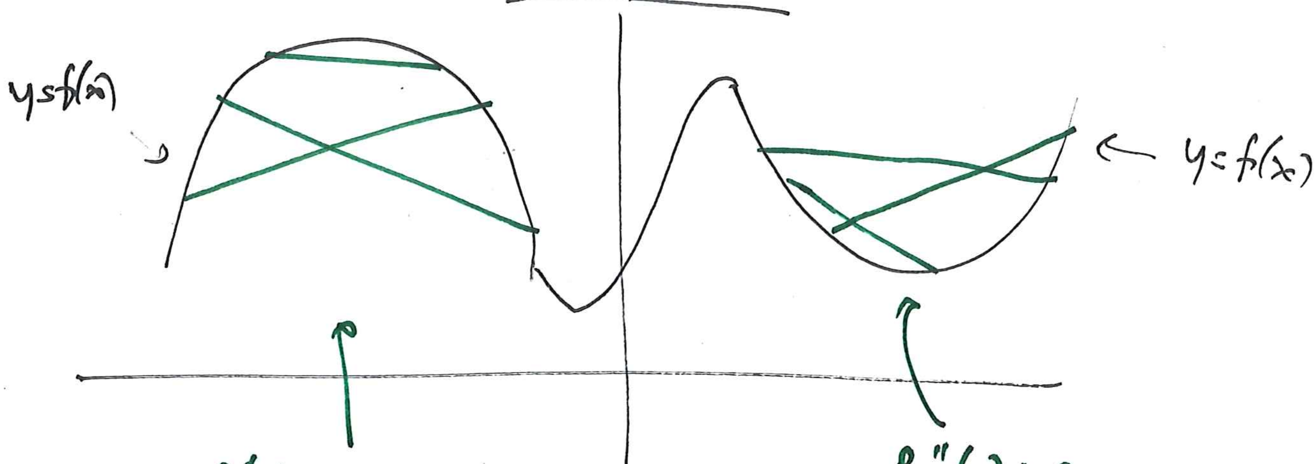
$$\Downarrow \qquad \text{(mult by } b-a>0\text{)}$$

$$f(b)-f(a) > 0$$

$$\Downarrow$$

$$f(b) > f(a)$$

New feature



$f''(x) < 0$

"f is concave down"

$f''(x) > 0$

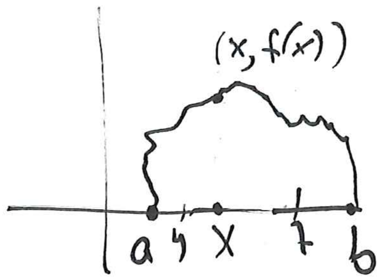
"f is concave up"

change in concavity called "inflection point".

2. THE SHAPE OF A THE GRAPH

(4) Let f be twice differentiable on $[a, b]$.

(a) Suppose first that $f(a) = f(b) = 0$ and that f is positive somewhere between a, b. Show that there is c between a, b so that $f''(c) < 0$.



f' exists everywhere, so by MVT have $a<y<x$ s.t. $f'(y) = \dfrac{f(x)-f(a)}{x-a} = \dfrac{f(x)}{x-a} > 0 \leftarrow \begin{matrix} f(x)>0 \\ x-a>0 \end{matrix}$

and have $x<z<b$ st $f'(z) = \dfrac{f(b)-f(x)}{b-x} = -\dfrac{f(x)}{b-x} < 0$

Now $y<x<z$. By MVT applied to f', have $y<c<z$ st:

$$f''(c) = \frac{f'(z)-f'(y)}{z-y} = \frac{1}{z-y}\left(f'(z) + (-f'(y))\right) < 0$$

(b) Now let $f(a), f(b)$ take any values, but suppose $f''(x) > 0$ on (a, b). Let $L : y = mx + n$ be the line through $(a, f(a)), (b, f(b))$. Applying part (a) to $g(x) = f(x) - (mx + n)$ show that the graph of f lies below the line L.

Summary of derivative info

$f'(x) > 0$ on $[a,b]$ $\Rightarrow$ f increasing there

$f'(x) < 0$ " " $\Rightarrow$ f decreasing ".

$f'(x) = 0$ or undef $\Rightarrow$ critical / singular pt.

$f''(x) > 0$ on $[a,b]$ $\Rightarrow$ f is concave up there

$f''(x) < 0$ " " " " " concave down ".

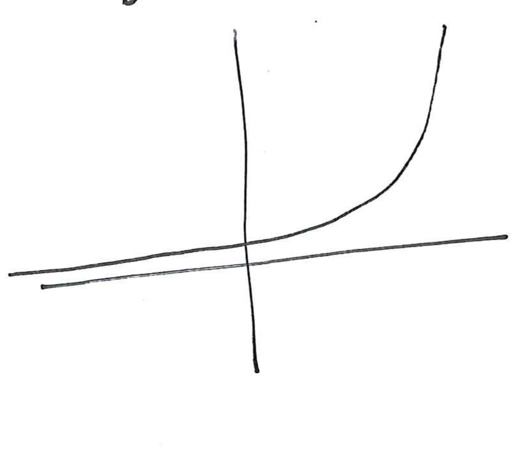
change between the two (where $f''(x) = 0$ or undef) is an inflection pt.

Example: 1.1) $f(x) = e^x$

0'th derivative: $f(x) > 0$ everywhere

1st derivative: $f'(x) = e^x > 0$, so f is increasing

2nd " : $f''(x) = e^x > 0$, so f is concave up.



Example: (2) $f(x) = \frac{x-7}{1+x^2}$

$$f'(x) = \frac{1+x^2}{(1+x^2)^2} - \frac{2x(x-7)}{(1+x^2)^2} = \frac{1+14x-x^2}{(1+x^2)^2} = \frac{50-(x-7)^2}{(1+x^2)^2}$$

$$f''(x) = \frac{-2(x-7)(1+x^2)}{(1+x^2)^3} - \frac{(50-(x-7)^2)\cdot 4x}{(1+x^2)^3} = \frac{2x^3-14x^2-6x+14}{(1+x^2)^3}$$

$f(x)>0$ if $x>7$ $\quad f(7)=0$

$f(x)<0$ if $x<7$

$f'(x)>0$ if $(x-7)^2<50$, i.e if $|x-7|<\sqrt{50}$,

i.e. if $7-\sqrt{50}<x<7+\sqrt{50}$

$f'(x)<0$ if $x<7-\sqrt{50}$, also if $x>7+\sqrt{50}$

Example: (3) $f(x) = \dfrac{x^2-9}{x^2+3}$

$f'(x) = \dfrac{24x}{(x^2+3)^2}$, $f''(x) = 72\,\dfrac{1-x^2}{(x^2+3)^3}$

(0) - where is f defined? cts? everywhere ($x^2+3>0$) so no problem
- where is $f>0$, $f<0$? $f>0$ on $(-\infty,-3)\cup(3,\infty)$
$f<0$ on $(-3,3)$, $f(-3)=f(3)=0$

- asymptotes?

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{1-9/x^2}{1+3/x^2} = 1$, same as $x\to-\infty$

(1) - where is f' defined? everywhere
- where is $f'>0$, $f'<0$? $f'(x)>0$ if $x>0$, $f'(x)<0$ if $x<0$ critical pt at $x=0$ (local min)

(2) - f'' defined everywhere

$f''>0$ if $-1<x<1$, $f''<0$ if $x<-1$, or $x>1$

$\Rightarrow$ ± 1 are inflection pts.

Summary

x	$(-\infty,-3)$	-3	$(-3,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,3)$	3	$(3,\infty)$
f	+	0	−	−	−	−	−	−	−	0	+
f'	−	−	−	−	−	0	+	+	+	+	+
f''	−	−	−	0	+	+	+	0	−	−	−