2. Limit laws (6/1/2017)

Goals.

(1) Limits, again
(2) Existence and nonexistence of limits: blowup
(3) Limit laws
(4) Limit "tricks"

Last Time.

(1) "limiting process" - in class: tangent lines
in book: instantaneous velocity

(2) limits: $\lim_{x\to a} f(x)$ is the value f would "like to have" at a.

Example: Consider $f(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

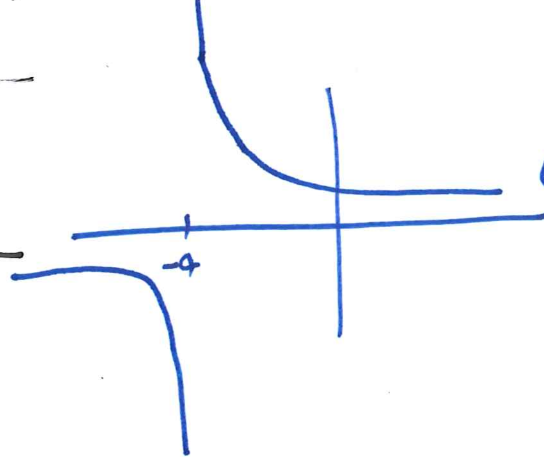
what is $\lim_{x\to 0} f(x) = ?$ $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$

near 0, $f(x) = x^2$

Example: $\lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$ ← got new expression for f made sense at $x=0$

$\frac{x}{x} = 1$ for $x \neq 0$

Worksheet (1)



← $y = \frac{1}{x+4}$

1. Existence of limits and blowup

(1) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

We have

$$\frac{x-3}{x^2+x-12} = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4} \xrightarrow[x\to 3]{} \frac{1}{7}$$

$$\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+4)} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{7}$$

DONT

$$\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{7}$$

(b) What about $\lim_{x\to 2} f(x)$? What about $\lim_{x\to -4^+} f(x)$, $\lim_{x\to -4^-} f(x)$?

$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{1}{x+4} = \frac{1}{6}$; At -4 $\lim_{x\to -4} f(x)$ DNE

does not exist

As $x\to -4$ from left, $x+4<0$, very small, so we say $\lim_{x\to -4^-} f(x) = -\infty$, When $x>-4$, $x+4>0$, so $\lim_{x\to -4^+} f(x) = +\infty$

(c) (Final, 2014) Evaluate $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

As $x\to -3$, $x+3\to 0$, for $x>-3$, close to -3, $x+2<0$, $x+3>0$ so $\frac{x+2}{x+3}<0$ so $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$.

Date: 10/9/2019, Worksheet by Lior Silberman.

(2) Evaluate

(a) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

for x close to 1, $(x-1)^2$ is smal and positive

so $\lim\limits_{x\to 1} \frac{1}{(x-1)^2} = +\infty$

(b) $\lim_{x\to\pi+} \frac{1}{\sin(x)}$, $\lim_{x\to\pi-} \frac{1}{\sin(x)}$.

if $x > \pi$, close to π then $\sin(x) < 0$

if $x < \pi$, close to π then $\sin(x) > 0$

also $\lim\limits_{x\to\pi} \sin x = 0$

so $\lim\limits_{x\to\pi^+} \frac{1}{\sin x} = -\infty$

$\lim\limits_{x\to\pi^-} \frac{1}{\sin x} = +\infty$

$\lim_{x\to a} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ exist if

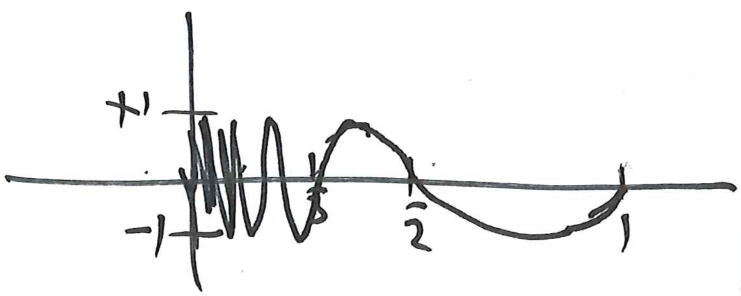
$f(x)$ approaches a definite value as $x \to a$ (as indicated)

otherwise the limit does not exist.

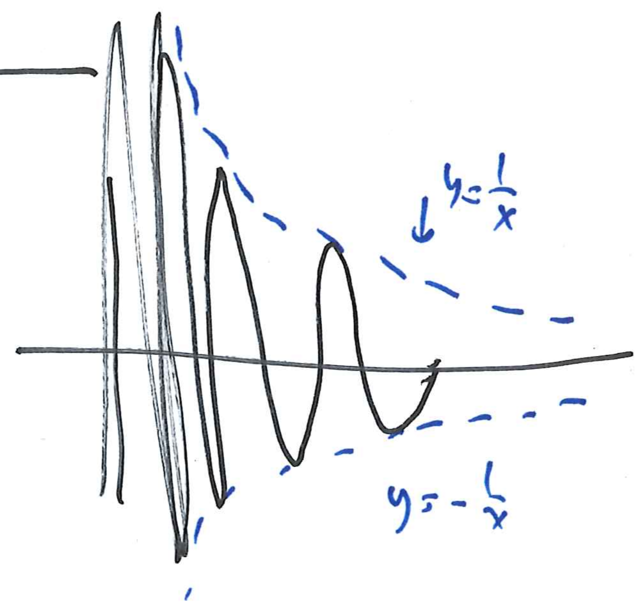
Work sheet (2)

Notice: need to know graph of $\sin x$, $\cos x$, $\tan x$

Example: $\lim_{x\to 0^+} \sin\left(\frac{\pi}{x}\right) = DNE$



Also, $\lim_{x\to 0^+} \frac{1}{x}\sin\left(\frac{\pi}{x}\right)$



One-sided limit DNE if function blows up or oscillates

Two-sided limit DNE if one-sided limits don't **or** if they exist but unequal

(3) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x^2) = 2-1^2 = 1$

so $\lim_{x\to 1} f(x) = 1$

(exists & equals 1)

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

Here, $\lim_{x\to 1^-} f(x) = 1$ (same reason)

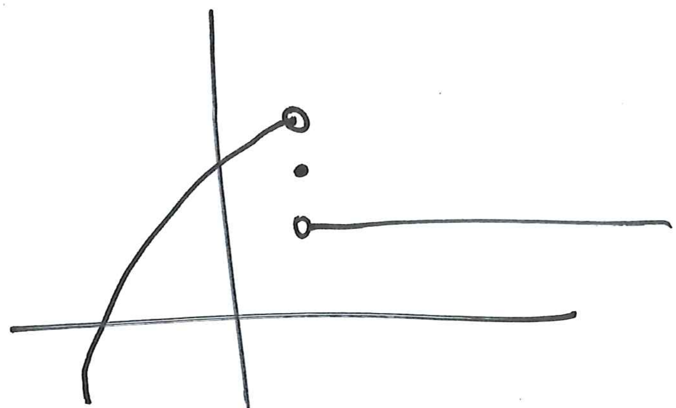
but $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4-x^2) = 4-1^2 = 3$

since the limits are different, $\lim_{x\to 1} f(x)$ DNE

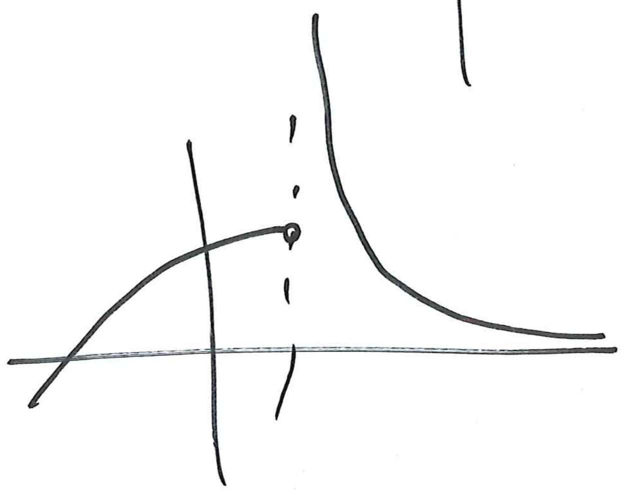
Example: $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ \frac{1}{x-1} & x > 1 \end{cases}$

Also $f(x) = e^{\tan x}$ as $x \to \frac{\pi}{2}$ (next lecture)

Examples



(both one-sided limits exist, unequal)



(limit exists on left, not on right)

2. Limit Laws

Fact. *Limits respect arithmetic operations and standard functions (e^x, sin, cos, log, ...)* **as long as everything is well-defined.**

(beware especially of division by zero)

(4) Evaluate using the limit laws:

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2-1} = \frac{3}{15} = \frac{1}{5}$

(b) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{e^x(x-1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{e^x}{x+2} = \frac{e}{3}$

Limit laws

Facts: (1) limits respect arithmetic.

If $\lim_{x\to 5} f(x) = 7$, $\lim_{x\to 5} g(x) = 2$

what is $\lim_{x\to 5} \frac{f(x)}{g(x)}$? it is $\frac{7}{2}$.

(2) limits of formulas well-behaved if formulas are.

Work sheet (2)

Two tricks for limits

(aside: compare $\lim_{x\to 0^+} \frac{x^2}{x}$, $\lim_{x\to 0^+} \frac{x}{x}$, $\lim_{x\to 0^+} \frac{x}{x^2}$)

consider $\sqrt{1+x} - 1 =$

$a^2 - b^2 = (a-b)(a+b)$

$$= (\sqrt{1+x} - 1)\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{(1+x) - 1}{\sqrt{1+x}+1} = \frac{x}{\sqrt{1+x}+1}$$

"rationalizing roots"

(5) Evaluate:

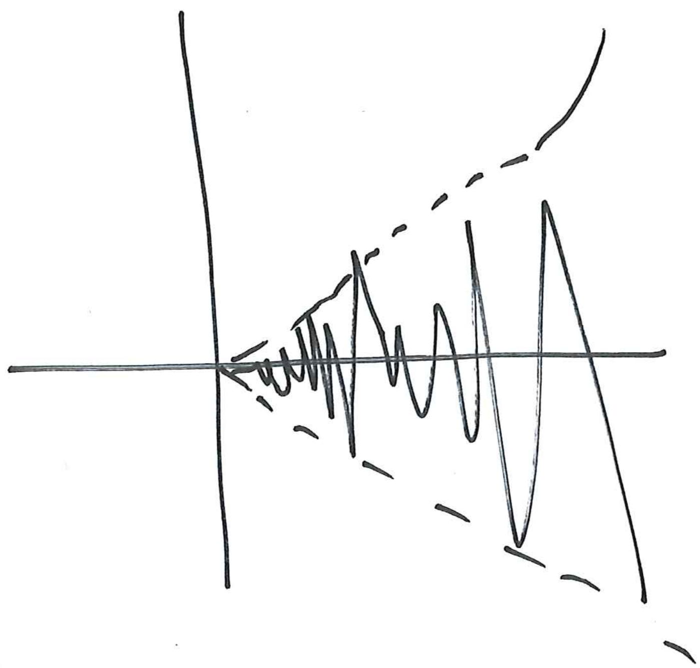
(a) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$.

$$\frac{\sqrt{4+x}-2}{x} = \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \frac{(4+x)-4}{x(\sqrt{4+x}+2)} = \frac{x}{x(\sqrt{4+x}+2)}$$

$$= \frac{1}{\sqrt{4+x}+2} \xrightarrow[x\to 0]{} \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$.

Squeeze thm



envelope of oscillation can force function to a limit.

Example: $f(x) = x^2 \sin\left(\frac{\pi}{x}\right)$