## Lior Silberman's Math 535, Problem Set 2: Representation Theory

### Basic constructions

1. In each case give a precise definition and show that reuslt is a continuous representation of the appropriate groups. In all cases $(\pi, V) \in \mathrm{Rep}(G)$.
   (a) $H \subset G$ a subgroup, $\mathrm{Res}_H^G \pi$ the *restriction* of $\pi$ to $H$ (still acting on $V$).
   (b) $W \subset V$ a closed subspace, $\pi \restriction_W$ the restriction of $\pi$ to $W$ (still a representation of $G$).
   (c) $W \subset V$ a closed subspace, $\bar{\pi}$ the representation of $G$ on $V/W$.
   (d) The representation $\pi \oplus \sigma$ of $G$ on $V \oplus W$ where $(\sigma, W) \in \mathrm{Rep}(G)$ also.

2. Consider the representation $\check{\pi}$ of $G$ on $V'$ where $\check{\pi}(g)\varphi = \varphi \circ \pi(g^{-1})$.
   (a) Show that $\check{\pi}(g) \colon V' \to V'$ is linear and that $\check{\pi}(gh) = \check{\pi}(g)\check{\pi}(h)$.
   (b) Show that $\check{\pi} \colon G \times V' \to V'$ is continuous where $V'$ is equipped with the weak-* topology (the locally convex topology determined by the seminorms $|\varphi|_{\underline{v}} = |\varphi(\underline{v})|$ where $\underline{v} \in V$
   (c) Show that $\check{\pi} \colon G \times V' \to V'$ is continuous where $V'$ is equipped with the *strong topology* (the locally convex topology determined by the seminorms $|\varphi|_E = \sup_{\underline{v} \in E} |\varphi(\underline{v})|$ where $E$ ranges over the bounded subsets of $V$).
   RMK If $V$ is a Banach space, the strong topology on $V'$ is exactly the norm topology with respect to the dual norm.

3. Let $(\sigma, W) \in \mathrm{Rep}(G)$
   (a) Show that the natural action $\pi \boxtimes \sigma$ of $G \times H$ on the algebraic tensor product $V \otimes W$ defines an action by linear maps.
   (b) Show that this action is a continuous representation if $V, W$ are finite dimensional.

### Constructions and Characters

Let $(\pi, V), (\sigma, W) \in \mathrm{Rep}(G)$ be finite-dimensional with characters $\chi_\pi, \chi_\sigma$..

4. We compute some characters.
   (a) Compute the characters of $\pi \oplus \sigma$, $\pi \otimes \sigma$ in terms of $\chi_\pi, \chi_\sigma$.
   (b) Let $U \subset V$ be $G$-invariant, and let $\tau(g) = \pi(g) \restriction_U$. Show that $\chi_{V/U} = \chi_\pi - \chi_\tau$.
   (c) Suppose instead that $\sigma$ was a representation of a group $H$ and compute the character of $\pi \boxtimes \sigma$ as a function on $G \times H$.

5. (Symmetric and antisymmetric tensor powers)
   (a) For $k \geq 2$ show that $\mathrm{Sym}^k V$, $\bigwedge^k V$ are $G$-invariant subspaces of $V^{\otimes k}$.
   DEF Write $\mathrm{Sym}^k \pi$, $\bigwedge^k \pi$ for the resulting representations.
   (b) Find the characters of $\mathrm{Sym}^k \pi$

### Examples of characters

6. Let $G$ be a finite group and let $X$ be a finite $G$-set.
   (a) Show that setting $(\pi(g)f)(x) = f(g^{-1} \cdot x)$ defines a linear representation of $G$ on $L^2(X)$ (counting measure).
   (b) Show that $\chi_\pi(g) = \#\mathrm{Fix}(g)$.

7. Consider the particular case of $G = S_n$ acting on $[n] = \{1, \ldots, n\}$.

(a) Show that $\pi \simeq \mathbb{1} \oplus V$ where $\mathbb{1}$ is the trivial representation on the constant vectors and $V$ is the orthogonal complement.

(b) Compute the character $\chi$ of the representation on $V$, verify that $\langle \chi, \chi \rangle_{L^2(S_n)} = 1$ and conclude that $\chi$ is irreducible.

(c) Use $L^2(G) \simeq \bigoplus_{\pi \in \hat{G}} \pi \boxtimes \check{\pi}$ to show that $\widehat{S_3} = \{1, V\}$ (hint: dimension count).

(c) Decompose the representation arising from the action of $S_n$ on the set $[n]^2$ into irreducibles, and connect it to problem 5.