Lior Silberman's Math 412: Problem set 8, due 9/11/2017

Practice: Norms

P1. Call two norms $\|\cdot\|_1, \|\cdot\|_2$ on V *equivalent* if there are constants $c, C > 0$ such that for all $\underline{v} \in V$,

$$c\|\underline{v}\|_1 \le \|\underline{v}\|_2 \le C\|\underline{v}\|_1 .$$

(a) Show that this is an equivalence relation.

(b) Suppose the two norms are equivalent and that $\lim_{n\to\infty} \|\underline{v}_n\|_1 = 0$ (that is, that $\underline{v}_n \xrightarrow[n\to\infty]{\|\cdot\|_1} \underline{0}$). Show that $\lim_{n\to\infty} \|\underline{v}_n\|_2 = 0$ (that is, that $\underline{v}_n \xrightarrow[n\to\infty]{\|\cdot\|_2} \underline{0}$).

(*c) Show the converse of (b) also holds. In other words, two norms are equivalent iff they determine the same notion of convergence.

Norms

1. Let $f(x) = x^2$ on $[-1,1]$.

(a) For $1 \le p < \infty$. Calculate $\|f\|_{L^p} = \left(\int_{-1}^{1} |f(x)|^p \, dx\right)^{1/p}$.

(b) Calculate $\|f\|_{L^\infty} = \sup\{|f(x)| : -1 \le x \le 1\}$. Check that $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_\infty$.

(c) Calculate $\|f\|_{H^2} = \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}$.

SUPP Show that the H^2 norm is equivalent to the norm $\left(\|f\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}$.

2. Let $A \in M_n(\mathbb{R})$. Write $\|A\|_p$ for its $\ell^p \to \ell^p$ operator norm.

(a) Show $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$.

(b) Show that $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$.

RMK See below on *duality*.

3. The *spectral radius* of $A \in M_n(\mathbb{C})$ is the magnitude of its largest eigenvalue: $\rho(A) = \max\{|\lambda| \mid \lambda \in \operatorname{Spec}(A)\}$.

(a) Show that for any norm $\|\cdot\|$ on $\mathbb{C}^n$ and any $A \in M_n(\mathbb{C})$, $\rho(A) \le \|A\|$.

(b) Suppose that A is diagonable. Show that there is a norm on $\mathbb{C}^n$ such that $\|A\| = \rho(A)$.

(*c) Show that if A is Hermitian then $\|A\|_2 = \rho(A)$.

(d) Show that if A, B are similar, and $\|\cdot\|$ is any norm in $\mathbb{C}^n$, then $\lim_{m\to\infty} \|A^m\|^{1/m} = \lim_{m\to\infty} \|B^m\|^{1/m}$ (in the sense that, if one limit exists, then so does the other, and they are equal).

(**e) Show that for any norm on $\mathbb{C}^n$ and any $A \in M_n(\mathbb{C})$, we have $\lim_{m\to\infty} \|A^m\|^{1/m} = \rho(A)$.

4. The *Hilbert–Schmidt* norm on $M_n(\mathbb{C})$ is $\|A\|_{\mathrm{HS}} = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$.

PRAC Verify that $\|A\|_{\mathrm{HS}} = \left(\operatorname{Tr}(A^\dagger A)\right)^{1/2}$.

(a) Show that $\|\cdot\|_{\mathrm{HS}}$ is, indeed, a norm.

(b) Show that $\|A\|_2 \le \|A\|_{\mathrm{HS}}$.

Extra credit: Norms and constructions

5. (Direct sum) Let $\{(V_i, \|\cdot\|_i)\}_{i=1}^n$ be normed spaces, and let $1 \le p \le \infty$. For $\underline{v} = (\underline{v}_i) \in \bigoplus_{i=1}^n V_i$ define
$$\|\underline{v}\| = \left(\sum_{i=1}^n \|\underline{v}_i\|_i^p\right)^{1/p} .$$
Show that this defines a norm on $\bigoplus_{i=1}^n V_i$.
DEF This operation is called the *L^p-sum* of the normed spaces.
6. (Quotient) Let $(V, \|\cdot\|)$ be a normed space, and let $W \subset V$ be a subspace. For $\underline{v} + W \in V/W$ set $\|\underline{v} + W\|_{V/W} = \inf\{\|\underline{v} + \underline{w}\| : \underline{w} \in W\}$.
   (a) Show that $\|\cdot\|_{V/W}$ is 1-homogenous and satisfies the triangle inequality (a "seminorm").
   (b) Show that $\|\underline{v} + W\|_{V/W} = 0$ iff v is in the closure of W, so that $\|\cdot\|_{V/W}$ is a norm iff W is closed in V.

For duality in norms see problems A, B. Norming tensor product spaces is complicated.

Supplementary problems: Constructions

A. Let V be a normed space. The operator norm on $V^* = \mathrm{Hom}_{\mathrm{cts}}(V, F)$ is called the *dual norm*.
   (a) Let $V = \mathbb{R}^n$ and identify V^* with $\mathbb{R}^n$ via the usual pairing. Show that the norm on V^* dual to the ℓ^1-norm is the ℓ^∞ norm and vice versa. Show that the ℓ^2-norm is self-dual.
   (b) Use A(a),(b) to show that the dual to the ℓ^p norm on $\mathbb{R}^n$ is the ℓ^q norm where $\frac{1}{p} + \frac{1}{q} = 1$.
   (c) Let U be another normed space and let $T: U \to V$ be bounded. Let $T': V' \to U'$ be the algebraic dual map as defined in this course. Show that for every $\underline{v}^* \in V^* \subset V'$, $T'\underline{v}^* \in U^*$ (that is, it is continuous). We write $T^*: V^* \to U^*$ for the dual map restricted to continuous functionals.
   (d) Show that T^* is itself bounded, in that $\|T^*\|_{V^* \to U^*} \le \|T\|_{U \to V}$.

B. A *seminorm* on a vector space V is a map $V \to \mathbb{R}_{\ge 0}$ that satisfies all the conditions of a norm except that it can be zero for non-zero vectors.
   (a) Show that for any $f \in V'$, $\varphi(\underline{v}) = |f(\underline{v})|$ is a seminorm.
   (b) Construct a seminorm on $\mathbb{R}^2$ not of this form.
   (c) Let Φ be a family of seminorms on V which is pointwise bounded. Show that $\bar{\varphi}(\underline{v}) = \sup\{\varphi(\underline{v}) \mid \varphi \in \Phi\}$ is again a seminorm.

Supplementary problem: Continuity

C. Let V, W be normed vector spaces, equipped with the metric topology coming from the norm. Let $T \in \mathrm{Hom}_F(V, W)$. Show that the following are equivalent:
   (1) T is continuous.
   (2) T is continuous at zero.
   (3) T is *bounded*: $\|T\|_{V \to W} < \infty$, that is: for some $C > 0$ and all $\underline{v} \in V$, $\|T\underline{v}\|_W \le C\|\underline{v}\|_V$.
   Hint: the same idea is used in problem P1

Supplementary problems: Completeness

D. Let $\{\underline{v}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in a normed space. Show that $\{\|\underline{v}_n\|\}_{n=1}^{\infty} \subset \mathbb{R}_{\geq 0}$ is a Cauchy sequence.

E. (The completion) Let (X,d) be a metric space.

(a) Let $\{x_n\},\{y_n\} \subset X$ be two Cauchy sequences. Show that $\{d(x_n,y_n)\}_{n=1}^{\infty} \subset \mathbb{R}$ is a Cauchy sequence.

DEF Let $(\tilde{X},\tilde{d})$ denote the set of Cauchy sequences in X with the distance $\tilde{d}\left(\underline{x},\underline{y}\right) = \lim_{n\to\infty} d\left(x_n,y_n\right)$.

(b) Show that $\tilde{d}$ satisfies all the axioms of a metric except that it can be non-zero for distinct sequences.

(c) Show that the relation $\underline{x} \sim \underline{y} \iff \tilde{d}\left(\underline{x},\underline{y}\right) = 0$ is an equivalence relation.

(d) Let $\hat{X} = \tilde{X}/\sim$ be the set of equivalence classes. Show that $\tilde{d} \colon \tilde{X}\times\tilde{X} \to \mathbb{R}_{\geq 0}$ descends to a well-defined function $\hat{d} \colon \hat{X}\times\hat{X} \to \mathbb{R}_{\geq 0}$ which is a metric.

(e) Show that $(\hat{X},\hat{d})$ is a complete metric space.

DEF For $x \in X$ let $\iota(x) \in \hat{X}$ be the equivalence class of the constant sequence x.

(f) Show that $\iota \colon X \to \hat{X}$ is an isometric embedding with dense image.

(g) (Universal property) Show that for any complete metric space (Y,d_Y) and any uniformly continuous $f \colon X \to Y$ there is a unique extension $\hat{f} \colon \hat{X} \to Y$ such that $\hat{f}\circ\iota = f$.

(h) Show that triples $(\hat{X},\hat{d},\iota)$ satisfying the property of (g) are unique up to a unique isomorphism.

F. (Complete fields) An *absolute value* on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ such that (a) $|xy| = |x|\,|y|$ (b) $|x| = 0 \leftrightarrow x = 0$ (c) $|x+y| \leq |x|+|y|$.

DEF Fix a prime number p. For $x \in \mathbb{Q}^{\times}$ write $x = \frac{a}{b}p^k$ for some non-zero $a,b \in \mathbb{Z}$ prime to p and $k \in \mathbb{Z}$ and set $|x|_p = p^{-k}$ (also, $|0|_p = 0$).

(a) Show that $|\cdot|_p$ is an absolute value on $\mathbb{Q}$ satisfying the *ultrametric inequality* $|x+y|_p \leq \max\left\{|x|_p, |y|_p\right\}$.

(b) Let $|\cdot|$ be an absolute value on F. Show that $d(x,y) = |x-y|$ is a metric on F.

(c) Show that (with respect to the metric of (b)) the absolute value is uniformly continuous $F \to \mathbb{R}_{\geq 0}$, addition is a uniformly continuous map $F\times F \to F$, and that for any $r > 0$ multiplication is a uniformly continuous map $B(0,r)\times B(0,r) \to B(0,r^2)$.

(d) Let $\hat{F}$ be the completion of F wrt $|\cdot|$. Show that the absolute value and the operations of addition and multiplication extend to maps $|\cdot| : \hat{F} \to \mathbb{R}_{\geq 0}$, $+,\cdot : \hat{F}\times\hat{F} \to \hat{F}$ giving it the structure of a ring with an absolute value $|\cdot|$.

(e) Show that every non-zero element of $\hat{F}$ has an inverse, that is that $\hat{F}$ is a field.

DEF Write $\mathbb{Q}_p$ for the completion of $\mathbb{Q}$wrt $|\cdot|$.

FACT Closed bounded sets in $\mathbb{Q}_p$ are compact, as in $\mathbb{R}$.