Math 322, lecture 12, 19/10/2017

Last time: Action of G on X = map $\cdot : G \times X \to X$
+ axioms

$\Leftrightarrow$ Homomorphism $G \to S_X$.

Question 2: Given action, to $g \in G$ we associated $\sigma_g : X \to X$ by $\sigma_g(x) = g \cdot x$.

fact: $\sigma_{gh} = \sigma_g \circ \sigma_h$

Saw: (1) $\sigma_g \in S_X$ (2) map $g \mapsto \sigma_g$ is a hom $G \to S_X$

Analogue: To $f \in C^1(\mathbb{R})$, set $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

facts: $(\alpha f + g)' = \alpha f' + g'$

$\Rightarrow$ map $D : C^1(\mathbb{R}) \to C(\mathbb{R})$ given by $Df = f'$ is linear.

Today: Conjugation

This will be an action of G on itself, but not the regular action

Def: For $g, x \in G$ set ${}^{g}x = \gamma_g(x) = g x g^{-1}$.

Lemma: (1) This is an action of G on itself

(2) It's an action by automorphisms: $\gamma_g \in \mathrm{Aut}(G)$

(3) $\gamma_\cdot : G \to \mathrm{Aut}(G)$ is a hom

Pf: Ex. (1) Need to show: ${}^{gh}x = {}^{g}({}^{h}x)$, ${}^{e}x = x$.

(2) $\gamma_g(xy) = \gamma_g(x)\gamma_g(y)$

(3) $\gamma_\cdot : G \to S_G$ is a hom, and we checked in (2) that image is contained in $\mathrm{Aut}(G)$

<u>Def</u>: Say $x, y \in G$ are <u>conjugate</u> iff there is $g \in G$ s.t. $y = gxg^{-1}$.

<u>Lemma</u>: This is an equivalence relation

<u>Pf</u>: PS3, Problem 2(a)

<u>Def</u>: The equivalence classes are ~~als~~ called the <u>conjugacy classes</u> in/of G. Sometimes the set of classes is denoted $G^{\natural}$

<u>Example</u>: - Conjugacy class of e is $\{e\}$: $geg^{-1} = e$ for all $g \in G$.

- conjugacy class of x is $\{x\}$ iff $x \in Z(G)$
  ($gxg^{-1} = x$ iff $gx = xg$)

<u>Remarks</u> Conjugacy is useful because (1) acts by <u>autos</u> (so conjugate elements ~~have~~ share group-theoretic properties)

(2) Readily available: no need for anything beyond G.

(<u>Def</u>: Image of $\gamma : G \to \mathrm{Aut}(G)$ is called the group of inner automorphisms, denoted $\mathrm{Inn}(G)$)

Ex: $\mathrm{Ker}(\gamma) = Z(G)$ so $\mathrm{Inn}(G) \cong G/Z(G)$.

Also, if $f \in \mathrm{Aut}(G)$, $g \in G$ then $f \circ \gamma_g \circ f^{-1} = \gamma_{f(g)}$

so $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$

Def: $\mathrm{Out}(G) \overset{\text{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$, called "outer automorphism group of G".

Example: $G = \mathbb{Z}^d$. Conjugacy is trivial since G is commutative: $g + x + (-g) = x$ for all g, x.

so $\mathrm{Inn}(\mathbb{Z}^d) = \{\mathrm{id}_{\mathbb{Z}^d}\}$ but $\mathrm{Aut}(\mathbb{Z}^d) = GL_d(\mathbb{Z})$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\{g \in M_n(\mathbb{Z}) \mid g^{-1} \in M_n(\mathbb{Z})\}$$

Example: If $\# X \geq 3$, then $Z(S_X) = \{\mathrm{id}\}$

so $\mathrm{Inn}(S_X) \cong S_X$

Fact: $\mathrm{Out}(S_n) = \{e\}$ except $\mathrm{Out}(S_6) \cong C_2$

if $f : S_n \to S_n$ $(n \neq 6)$ is an isom then $\exists \rho$ s.t. $f(\sigma) = \rho \sigma \rho^{-1}$.

Lemma: There is a bijection between the conjugacy class of $x \in G$ and the coset space $G/Z_G(x)$.

Cor: The number of conjugates of x is $[G:Z_G(x)]$

Pf: Map $gZ_G(x) \mapsto {}^g x = gxg^{-1}$.

(0) well-defined: if $g' = gz$, $z \in Z_G(x)$ then ${}^{g'}x = g'x(g')^{-1} =$

$$= (gz)x(gz)^{-1} = g(zxz^{-1})g^{-1} \underset{\substack{\uparrow \\ z \in Z_G(x)}}{=} gxg^{-1} = {}^g x$$

(1) Surjective: gxg^{-1} is image of $gZ_G(x)$

(2) injective: if ${}^g x = {}^{g'}x$ then ${}^{g^{-1}g'}x = {}^{g^{-1}g}x = {}^e x = x$,

so $gxg^{-1} = g'x(g')^{-1}$

$$\Downarrow$$

$$x = g^{-1}g'x(g')^{-1}g = (g^{-1}g')x(g^{-1}g')^{-1}.$$

so $g^{-1}g' \in Z_G(x)$, so $gZ_G(x) = g'Z_G(x)$ ∎

Thm: (Class equation) Let G be finite. Then

$$\#G = \#Z(G) + \sum_{\{x\}} [G:Z_G(x)]$$

where the sum is over the non-central conjugacy classes

PF: G is the disjoint union of its conjugacy classes

Note: Every summand divides $\#G$.

Ex: Interpret class equation as a combinatorial identity when $G = S_X$

Review: conjugacy of subgroups

Def: For $g \in G$, $H < G$ set ${}^gH = gHg^{-1} = \gamma_g(H)$

Lemma: This is an action of G on its set of subgps.

Pf: Same as before: ${}^eH = eHe^{-1} = H$

and ${}^{gh}H = (gh)H(gh)^{-1} = ghHh^{-1}g^{-1} =$
$= g(hHh^{-1})g^{-1} = {}^g({}^hH)$.

Def: Call $H, K < G$ conjugate if ${}^gH = K$ for some $g \in G$.

Lemma: This is an equivalence relation.

Example: The class of H is $\{H\}$ iff H is normal

Lemma: There is a bijection between class of H and $G/N_G(H)$

Pf: Same: map $gN_G(H) \mapsto gHg^{-1}$

(0) well-def: if $n \in N_G(H)$, then $(gn)H(gn)^{-1} = g(nHn^{-1})g^{-1} = gHg^{-1}$.

(1) surjective: gHg^{-1} = image of $gN_G(H)$

(2) injective: if $gHg^{-1} = hHh^{-1}$ then $(g^{-1}h)H(g^{-1}h)^{-1} = H$

so $g^{-1}h \in N_G(H)$ and $gN_G(H) = hN_G(H)$

Second Review: General actions

Fix group G acting on a set X.

Def: Say $x, y \in X$ are <u>in the same orbit</u> if $\exists g \in G : g \cdot x = y$

Lemma: This is an equivalence relation

Pf: $e \cdot x = x$ so x, x are in same orbit.

If $g \cdot x = y$ then $x = g^{-1} \cdot (gx) = g^{-1} y$ so y, x are in same orbit.

If $y = g \cdot x$, $z = h \cdot y$ then $z = h \cdot (g \cdot x) = (hg) \cdot x$.

Def: Equivalence classes are called the ~~orbit~~. <u>orbits</u> of G on X.

Write $G \cdot x$ or $\mathcal{O}(x)$ for the orbit of $x \in X$,

write $G \backslash X$ for the set of orbits

<u>Example:</u> (Poincaré): X = phase space of solar system

point $x \in X$ = specifying positions and velocities of all planets

Action $(\mathbb{R}, +)$ on X: $g_t \cdot x = y$ where if we start system at point x at time 0, it reaches y at time t.