Math 322, lecture 10, 10/10/17

Last time: N normal $\Rightarrow$ G/N is a group with $gN \cdot hN = ghN$

$q: G \to G/N$ quotient map.

Today: (1) Isom thms

(2) A_n is simple if $n \geq 5$

Thm: let $f \in \mathrm{Hom}(G, H)$, and let $K = \mathrm{Ker}(f)$. Then f induces an isom $\bar{f}: G/K \longrightarrow \mathrm{Im}(f)$

diagram:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "q"'] & \mathrm{Im}(f) \subset H \\ & G/K \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$

Pf: Define $\bar{f}(gK) = f(g)$ (well-defined since if $gK = g'K$ then $g' = gk$ for some $k \in K$, so $f(g') = f(gk) = f(g)f(k) = f(g)$ since $k \in K = \mathrm{Ker}(f)$)

Next, $\bar{f}((gK)\cdot(hK)) \underset{\text{def of } G/K}{=} \bar{f}(gh \cdot K) \underset{\text{def of } \bar{f}}{=} f(gh) \underset{f \text{ is a hom}}{=} f(g)f(h) \underset{\text{def of } \bar{f}}{=} \bar{f}(gK)\,\bar{f}(hK)$.

The image of $\bar{f}$ is that of f by construction.

[Ex: a gp hom is injective iff its kernel is trivial]

Finally, $\mathrm{Ker}(\bar{f}) = \{gK \mid \bar{f}(gK) = e_H\} \overset{\text{def of } \bar{f}}{=} \{gK \mid f(g) = e_H\} =$

$\underset{K = \mathrm{Ker}(f)}{=} \{gK \mid g \in K\} = \{K\} = \{e_{G/K}\}$,

so $\bar{f}$ is injective.

Call this "1st isomorphism thm".

(Use this: Know G, K want to "know" G/K
make hom $f: G \to H$ s.t. $\ker(f) = K$
perhaps $\mathrm{Im}(f)$ is easier to understand)

(Example: G, H vector spaces, $f: G \to H$ linear map.
Thm: $G/\ker(f) \cong \mathrm{Im}(f)$ i.e. $\dim G - \dim \ker(f) = \dim \mathrm{Im}(f)$)

Thm: (2nd isom thm) Let $N, H < G$ with N normal.
Then $N \cap H \triangleleft H$ and the map $H \to HN$ induces an
isom $H/N \cap H \cong HN/N$

$HN = \{hn \mid h \in H, n \in N\} < G$
proved in PS 5

(start with HN, "kill off" N. left with H, but
with elements of $N \cap H$ "killed off", i.e. with $H/N \cap H$)

Pf: Let $\iota: H \to HN$ be the inclusion map $\iota(h) = h$
Let $\pi: HN \to HN/N$ be the ~~restriction of~~ the quotient map.
(HN is a subgp of G; $N \subset HN$ so a subgp there; $N \triangleleft HN$ since
$N \triangleleft G$: $\forall g \in HN$ have $gNg^{-1} = N$)

Let $f: H \to HN/N$ be the composition $\pi \circ \iota$, i.e.
$f(h) = hN$.

f is surjective. for any $(hn)N \in HN/N$ we have:

$hnN = h(nN) = hN = f(h)$

Also,

$Ker(f) = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N.$

Since $H \cap N = Ker(f)$, it is normal in H (domain of f).

By the 1st isom thm, f induces an isom

$$\bar{f} : \underbrace{H/H \cap N}_{Ker(f)} \xrightarrow{\simeq} \underbrace{HN/N}_{Im(f)}$$

<u>Thm</u>: (3rd isom thm) Let $K < N < G$ be subgps with both K, N normal in G. Then $N/K \trianglelefteq G/K$ and we have an isom $(G/K)/(N/K) \simeq G/N$.

<u>Pf</u>: Let $f : G/K \to G/N$ be given by $f(gK) = gN$.

(f is well-defined: if $gK = g'K$, i.e. if $g^{-1}g' \in K$ then $g^{-1}g' \in N$ ($K \subset N$) so $gN = g'N$)

f is a gp hom: $f((gK)\cdot(hK)) \underset{\substack{\text{def of } \cdot \\ \text{in } G/K}}{=} f(ghK) = ghN \underset{\substack{\text{def of } \cdot \\ \text{in } G/N}}{=} gN \cdot hN = f(gK) f(hK)$

f is surjective: any coset gN arises (as $f(gK)$)

$Ker(f) = \{gK \mid gN = N\} = \{gK \mid g \in N\} = N/K$. It follows that $N/K \trianglelefteq G/K$ and, by 1st isom thm, that $(G/K)/(N/K) \simeq G/N$.

Example: $GL_2(\mathbb{R})/SL_2(\mathbb{R}) \cong \mathbb{R}^\times$

$GL_2(\mathbb{R})$: 2×2 invertible matrices

$SL_2(\mathbb{R})$: same, det = 1

$SL_2(\mathbb{R}) = \ker(\det)$, $\det : GL_2(\mathbb{R}) \to \mathbb{R}^\times$

by 1st isom thm, $GL_2(\mathbb{R})/\ker(\det) \cong \operatorname{Im}(\det) = \mathbb{R}^\times$

Correspondence is

$$GL_2/SL_2 \ni g\,SL_2(\mathbb{R}) \leftrightarrow \det(g) \in \mathbb{R}^\times$$

Simplicity of A_n, $n \geq 5$

Recall **Def:** G is *simple* if its only normal subgps are $\{e\}$, G.

Lemma: The pairs $(123), (145)$, and $(12)(34)$, $(12)(35)$ are conjugate in A_5, hence in A_n, $n \geq 5$.

Pf: Conjugate by $(24)(35)$ and (345) respectively.

Lemma: (1) All 3-cycles are conjugate in A_n, generate it.

(2) All elements $\tau_1\tau_2$ with τ_i disjoint transpositions are conjugate in A_n, generate it.

Pf: PS 3

Thm: A_n is simple if $n \geq 5$.

Pf: Let $N \triangleleft A_n$ be normal, $N \neq \{e\}$.

Let $\sigma \in N \setminus \{id\}$ have minimal support.

Goal: show σ = 3-cycle or pdt of two disjoint transpositions.

In either case, N contains all conjugates of σ (N is normal, and all conjugates together generate A_n (lemma)

so $N = A_n$.

For this: wlog support of σ is $\{1, 2, \dots, k\}$.

Case 1: $k=1$. Impossible: $\sigma = id$

Case 2: $k=2$ " : σ = transposition

Case 3,4: $k=3 \Rightarrow \sigma$ is a 3-cycle

$k=4 \Rightarrow \sigma$ is a pdt of transp (not 4-cycle, those are odd)

$k=5$, $k \geq 6$: next time.